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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION

Don't Drink the Kool-Aid®!

Kristin Burney, Lydia Kennedy, Audrey Malagon
Virginia Wesleyan College
Norfolk VA 23502 USA

Abstract: Single-compartment mixing is an important foundational component of any study of ordinary differential equations. Typically, problems utilize salt as the solute. In this modeling scenario, use of colored drink powder as the solute enables students to observe a color change as the mixing progresses. Students run two experiments that involve a concentrated solution of drink powder flowing from an upper tank to a lower tank, initially filled with plain water. They make qualitative observations about the intensity of the color in the tanks and record initial conditions. In the first experiment, the spout on the lower tank remains closed; in the second, the solution flows from both tanks at the same flow rate. Prior to experimentation, students are asked to predict the outcomes and sketch results. Following experimentation, students develop mathematical models and use them to determine the amount of drink powder in the lower tank at the end of each experiment. Specifically, first-order separable and linear differential equations are employed. Additional components of the lab invite students to consider a situation in which tank outflow rates differ (i.e., changing tank volume over time) and, further, a scenario in which pollution is entering and leaving a lake via streams.

SCENARIO DESCRIPTION

For this project, we will explore how the amount of drink powder in a tank changes when varying concentrations of powder and water solution are added to a tank that begins with pure water. As you perform these experiments, keep careful notes, recording any and all data and observations as well as answers to the questions in each part. You will need these to write your final lab report. To set up, stack one tank above the other as shown in Figure 1, with a container under the bottom tank to catch the water flowing out.

MATERIALS

The experiments should be completed in a lab environment with access to water (and paper towels!).

Figure 1. Set up for data collection.

- 2 drink dispensers (approximately 8 liters or larger)
- riser (for elevating one of the drink dispensers)
- collecting bucket
- balances (One for every two set-ups is sufficient.)
- bright-colored drink powder (Each group will need at least 100 grams.)
- timer (A cell phone app is sufficient.)
- pitcher (something to help fill the dispensers)
- mixing stick (Paint-mixing sticks work great.)
- tape and markers (if dispensers are not pre-marked)

Part I. Setting the stage

For our experiments, we will assume that the flow rate out of the top tank and into the bottom tank is the same as the flow rate out of the bottom tank. We will also assume that the flow rate is constant throughout the experiments. If the flow rates are the same and constant, what should happen if the spouts on both tanks are opened fully at the same time?

Let's first test that these assumptions are reasonable by determining the flow rate out of each tank; for this, you will need a timer, such as a stopwatch app. Fill both tanks with water to the same level, making note of the water level. When your timer is ready, open both spouts simultaneously and watch what happens. Describe the results of this test and whether or not our assumptions are reasonable. Also calculate and make a note of what the flow rate is. Adjust the assumptions if necessary, and make notes about when the assumptions are valid (i.e., until the tank reaches a certain point) or any modifications you had to make to the process, e.g., holding one spout slightly more open. When you have finished performing this test, CLOSE THE SPOUTS and refill your tanks!

Part II. One spout open

Now we will make a solution of water and drink powder in the top tank. We want this to be fairly concentrated, so use enough to make your mixture a nice vibrant color. Use the balances to carefully measure and record the amount of drink powder in grams that you are adding to the water. Make sure that ONLY the top tank contains the drink powder mixed with water. The bottom tank should have half the volume of the top tank, but initially be plain water.

Before you let any liquids flow, describe what you expect to happen to the liquid in the bottom tank, which starts out as pure water. (Will it end as pure water? as powder-water mixture? Will it have the same concentration as the top tank? How long will it take for your predicted outcome to occur?) Sketch a graph of your prediction of the amount of powder-water mixture in the bottom tank at time t.

Let it flow! Open the spout on the top tank ONLY and watch what happens, recording any significant observations, including, but not limited to, timing how long you run your experiment. How long is a reasonable amount of time to conduct the experiment? Is there a limit on how long you can run the experiment?

Later you will create a model for the amount of drink powder A in the bottom tank at time t, discuss what your model predicts in the short and long term, and use this model to determine the amount of drink powder (in grams) that is in the bottom tank at the end of your experiment. The differential equation that you start with should consider the rate of change of amount of drink powder, $\frac{dA}{dt}$. Write down any information NOW that you will need to write this model later. If time allows, you may work on this model before going on to the next part, but be sure to leave yourself enough time to do the next part. Note: For your model, you may assume that mixing is instantaneous.

Part III. Both spouts open

This time we will again let the mixture flow from the top tank into the bottom tank, but we will start with plain water in the bottom tank of the SAME volume as the top tank. Carefully create your powder-water mixture, recording the amount of drink powder used in grams. Perform the experiment again, this time opening BOTH spouts simultaneously, and watch what happens. Make sure the collection container is in place below the bottom tank. How do you need to adjust your model for this scenario? How much drink powder A do you think is in the second tank when the experiment ends? Is there a limit on how long you can run the experiment?

Modify your model to fit this second scenario and use it to answer the question of how much drink powder A is in the second tank at the end of the experiment. Does your model predict what you expected? Discuss.

Part IV. Varying outflow rates

(Modeling only; do not do this experiment.) At time $t = 0$, a tank contains $A(0) = 4$ grams (g) of drink powder dissolved in 100 liters (L) of water. Water containing 0.25 g of drink powder per liter is entering the tank at a rate of 3 L/minute, and the well-stirred solution leaves the tank at a rate of 3.5 L/minute.

a) Build a differential equation $\frac{dA}{dt}$ for the amount of drink powder A in grams that is in the tank at time t in minutes.
b) Find an expression for the amount of drink powder, $A(t)$, in grams that is in the tank at time t and plot $A(t)$ vs. t over the time interval $[0, 200]$ minutes.
c) Determine when the amount of drink powder doubles from the original amount in the tank.
d) Determine when the amount of drink powder in the tank is 20 g.
e) Determine when the amount of drink powder in the tank is 30 g.
f) Determine the maximum amount of drink powder in the tank and when it occurs.
g) Describe the long-term behavior of the amount of drink powder in the tank using accompanying plots to support your description.

Part V. Real-world application

(Modeling only; do not do this experiment.) Mixing problems are often used to create models for pollution entering and leaving lakes through streams. Describe a scenario like this, and create a differential-equations model. State all assumptions and describe what your model predicts in the long term. You may consult outside resources to create a realistic model, but you should cite them. You may not take differential equations from outside sources without fully explaining them.

WRITE UP

Create a lab report that describes your models, assumptions, data, and solutions for each of the four scenarios (Parts II-V) in the following format. For each scenario give these sections:

1. **Background**. Here you should include your assumptions, set up, and predictions and any changes you made to your assumptions after running the experiment.

2. **The Model**. Here you should give the mathematical model, a justification for why this is an appropriate model, and a qualitative analysis of the differential equation (Examples are: What does the model predict in the long term? What are the equilibrium solutions?) Are there any restrictions on your model, i.e., is it only valid when $t > 0$ or $A < 1000$?
3. **Solutions**. Give an analytic solution (if possible), plot the slope field with a graph of your solution on the slope field, and give a numerical approximation if an analytic solution is not available.