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A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS
& OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

STUDENT VERSION
SPREAD OF OIL SLICK

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Abstract: We describe a modeling activity for Calculus I students in which modeling with difference and differential equations is appropriate. This model enlightens students as to how derivatives are used in applications as well as a brief introductory encounter with parameter estimation for a linear, first-order, non-homogeneous, ordinary differential equation. This first encounter Calculus I activity focuses on connecting Calculus concepts with the real world using a guided/templated approach. A more open-ended approach such as "knowing what to do when you do not know what to do." [**?**, **?**] could perhaps be adopted in some later class or next course.

SCENARIO DESCRIPTION

You work for an environmental company. An oil slick is spreading at sea. Your job is to estimate how many ships and other materials will be needed to clean up the oil. This activity will guide you in doing the following steps: (1) collecting data on the size of the oil slick, (2) reviewing basic derivatives (3) determining a mathematical model for the size of the slick, (4) solving differential equations, (5) using model to predict the size of the oil slick after two weeks and estimating the amount of materials and the cost of cleanup.

Although **some photos** (https://qz.com/1577099/pictures-from-the-1989-exxon-valdez-disaster-30-years-ago/) were provided by the local forestry department, they are not exactly what you need for predicting the size of the oil slick. You need areal photos so that you can estimate the area, in square miles, of the oil slick. Luckily you are a skilled pilot and your small business has a company helicopter you can use any time without much fuss or pesky paperwork.

Due to your work and family schedule you manage to get up into the helicopter at irregular times. On each trip, you fly over the oil slick, take a photo, **wait 10 minutes**, then take a second photo, and then head home. Each photo is used to determine the area (in square miles) of the slick.

After taking all those photos, unfortunately you realize the data on your camera is corrupt and the dates/times are not appearing in the photo metadata. Since there was no log of your trips in the company paperwork, the best you can do is order and group the photos by trip number. The table below shows your results.

Area of Slick (in square miles)	
Trip 1, # 1	1.047
Trip 1, # 2	1.139
Trip 2, # 1	2.005
Trip 2, # 2	2.087
Trip 3, # 1	3.348
Trip 3, # 2	3.413
Trip 4, # 1	5.719
Trip 4, # 2	5.765
Trip 5, # 1	7.273
Trip 5, # 2	7.304
Trip 6, # 1	8.410
Trip 6, # 2	8.496
Trip 7, # 1	9.117
Trip 7, # 2	9.127
Trip 8, # 1	
Trip 8, # 2	

Table 1. Data on the size of the oil slick.

PART 1. DATA COLLECTION

(a) In order to make your project unique, fill in values for Trip 8 above. Make sure they are reasonably in accordance with previous values. Your values should be slightly different from every other student.

(b) In the table below, calculate the average rate of change of the area of the oil slick for each individual trip. Remember that average rate of change is represented by the slope of the secant line formula here as $\frac{A_2-A_1}{t_2-t_1}$. Discuss with your peers how you obtain $A_2 - A_1$ for the numerator and how you obtain $t_2 - t_1$ for the denominator. Notice that the first entry of the table has been filled in for you, and the left column follows a certain pattern you should figure out.

Size of Slick (in square miles)		
Trip #	A	$A' \approx \frac{A_2-A_1}{t_2-t_1}$
1	1.047	0.0092
2	2.005	
3	3.348	
4	5.719	
5	7.273	
6	8.410	
7	9.117	
8		

Table 2. Data on the area of the oil slick, and the average rate of change.

PART 2. REVIEW DERIVATIVES & INTRO TO DIFFERENTIAL EQUATIONS

A *differential equation* is an equation that expresses a relationship between a function y and its derivative y'. The most basic differential equation is

$$y = y'.$$

In order to *solve* a differential equation, it means to *find a function* that satisfies the differential equation. What is a function that has the property that the original function y is equal to its own derivative y'?

$$y(x) = _____$$

Congratulations, you have now *solved* your very first differential equation!!
Below we will explore more differential equations.

Basic Derivatives & Differential Equations.

(a) What is the derivative of $y(x) = e^{2x}$?

$$y'(x) = _____$$

For this function, what number completes the differential equation below?

$$y' = ___ \, y$$

(b) What is the derivative of $y(x) = e^{-3x}$?

$$y'(x) = _____$$

For this function, what number completes the differential equation below?

$$y' = ___ \, y$$

(c) What is the derivative of $y(x) = e^{0.109x}$?

$$y'(x) = _____$$

For this function, what number completes the differential equation below?

$$y' = ___ \, y$$

(d) What is the derivative of $y(x) = e^{kx}$ where k is a constant?

$$y'(x) = _____$$

For this function, what constant completes the differential equation below?

$$y' = ___ \, y$$

PART 3. BUILDING A MODEL

Using **the Curve Fit website** (https://mycurvefit.com/) or **the Planet Calc website** (https://planetcalc.com/8735/) take your data in Table 2 find an *equation of a line* that best fits the data where the parameters (the slope and y-intercept) are estimated/rounded to seven decimal places. Rather than letters x, y use letters that are appropriate for this problem A, A'.

This equation is a differential equation because it gives a relationship between a function A and its derivative A'. We will solve this new differential equation in a series of steps below. Note: We recognize that $=$ could just as easily, and more properly, be replaced by $\sim$, a close approximation, since the values of A' were estimated.

PART 4. SOLVING THE DIFFERENTIAL EQUATION

Previously in PART 2, we solved the differential equation $y' = ky$, but the new differential equation in PART 3 has the general format of $y' = ky + b$ where k and b are constants. We must therefore expand our horizons and try to think of new functions $y(x)$ to solve the new differential equation.

Let's consider a similar differential equation $y' = 2y + 3$. We will solve this differential equation by guess and check. Our goal is to find a function $y(x)$ where the first derivative y' is *equal to* the quantity $2y + 3$.

(a) Define function $y(x) = 3e^{2x}$.

What is the derivative?

$$y'(x) = _____$$

Using the given original function simplify the expression.

$$2y(x) + 3 = _____$$

Is y' equal to $2y + 3$? Circle YES or NO.

(b) Let's try again. Define function $y(x) = e^{2x} - 3$.
What is the derivative?

$$y'(x) = _____$$

Using the given original function simplify the expression.

$$2y(x) + 3 = _____$$

Is y' equal to $2y + 3$? Circle YES or NO.

(c) Let's try again. Define function $y(x) = Ce^{2x} - \frac{3}{2}$ where C is a constant.
What is the derivative?

$$y'(x) = _____$$

Using the given original function simplify the expression.

$$2y(x) + 3 = _____$$

Is y' equal to $2y + 3$? Circle YES or NO.

(d) Now let's return to the differential equation from PART 3 which has the general format of $y' = ky + b$ where k and b are constants. Based on what we learned in (a)-(c) above, what do you think is the solution? Your answer should be a *function of* x, involving constants k, b and C.

$$y(x) = _____$$

Check your answer below by verifying that $y'(x)$ is indeed equal to $ky(x) + b$. Show the work.

(e) Finally, find what function $A(t)$ solves *your* differential equation from PART 3 using *your specific* k and b values. (Again, determine any constants you have to 7 decimal places.). Notice that here you should have A as function of t, not y as a function of x. Your function should also have a constant C in it.

$$A(t) = _____$$

Finally, use the initial measurement $A(0) = 1.047$ to find your C-value.

Again, state your function, but now include your C-value.

Your function is now ready to make predictions!

PART 5. MAKING PREDICTIONS

(a) Use your function from PART 4 to predict the size of the oil slick, at $t = 500$ min. How many minutes does this correspond to?

(b) Provide a sketch of the plot of your predictive function $A(t)$ of the size of the oil slick as a function of time. Notice there is a horizontal asymptote in the graph that should be correctly labeled. Also make sure your initial value of $A(0) = 1.047$ is labeled on your picture.

(c) If nobody cleans up the oil, what is the area of the oil slick as $t \to \infty$?
In other words, find $\lim_{t\to\infty} A(t)$.

(d) Find how many minutes after your initial observation the oil slick will be 8 square miles. Show your work.

(e) Find the t-values (in minutes) when your function $A(t)$ predicts that the second and the third measurements were made. How long after Trip #1 was Trip #2 made?

(f) Suppose that it takes you exactly 3 days after your initial observation to hire all the ships you can manage to hire to help clean up the spill. Assume that one ship can clean up two square miles of oil. How many ships will you need to hire in order to eliminate the oil slick at day 3? If one ship costs $ 20,000 to hire, how much will the cleanup cost at day 3?

(g) Brainstorm some ways in which solving the oil slick problem may differ in real life. How would this translate into different equations and/or different functions? Provide at least 2 new ideas that you would encounter in a real life version of this scenario. Don't just explain problems or difficulties, but also suggest a possible way your idea would lead to different equations or solutions in this lab. NOTE: This is more about the process of brainstorming than "being right". Express your answers in clearly written complete sentences.