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STUDENT VERSION

Theoretical and numerical study of chlorine concentration dynamics in regular swimming pool maintenance cycles

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Abstract: This project involves the dynamics of chlorine concentration during regular swimming pool maintenance cycles. Students will have the opportunity to use both analytic and numerical methods. On the analytical side, students will solve one of the model equations, describing the first stage of a two-stage mixing process, by separating variables, and the other by applying the method of integrating factors. On the numerical side, students will apply Euler's method to solve the model equations numerically, with the opportunity to examine numerical errors for various step sizes, while comparing results to the previously obtained exact solutions.

SCENARIO DESCRIPTION

Background

The concentration of chemical agents in pool water is regulated by health and safety standards to ensure safe use of recreational facilities [2]. Chlorine is one of the most common disinfectants added to pool water in order to suppress the growth of pathogenic microorganisms [1]. This project is concerned with the concentration of chlorine in a mid-sized swimming pool during a regular maintenance cycle.

A certain swimming pool is 20m long, 8m wide, and 2m deep. When filled, it contains a volume of $84,000$ gallons of water. Recommended levels of chlorine concentration range from 1.3×10^{-5} to 2.6×10^{-5} ounces per gallon of water. The pool's regular maintenance schedule involves filtering, cleaning, and refilling the pool at regular time intervals, while adhering to recommended health and safety standards.

A particular incident during one of the regular maintenance cycles can be described by a two-step process as follows.

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Stage 1: Initial refill. As a result of a miscalculation, water of a specified concentration of 1×10^{-7} ounces of chlorine per gallon of water is supplied at a rate of 5 gallons per minute into the swimming pool, which initially had a concentration of 1.3×10^{-5} ounces of chlorine per gallon of water. The mixture is well-stirred in the pool. A water pump is turned on to filter and heat the pool water, so that the well-stirred mixture drains at the same rate.

Stage 2: Post-quality-check refill. After 24 hours, chlorine levels are found to be insufficient, and water of an increased concentration of 5×10^{-5} ounces of chlorine per gallon of water is now supplied at a rate of 5 gallons per minute, with the well-stirred mixture draining from the pool at the same rate.

This project consists of two parts. Part I is a group activity, which you should complete in groups of 2-3. Part II is an individual project which you should complete yourself.

Group activity - Part I

Form a group of 2-3. Group members:

_____ _____ _____

Complete the following with your group.

1. Model formulation – Stage 1
   (a) Denote the total amount of chlorine in the pool at time t by $Q(t)$ (in ounces). Write down an expression denoting the rate of change of amount of chlorine.
   (b) Formulate a differential equation modeling the amount of chlorine, $Q(t)$, during Stage 1. Define any new constants, if necessary.
   (c) Write down the initial condition(s).
2. Separate variables to solve the differential equation modeling Stage 1. In particular:
   (a) Obtain a general solution to the differential equation
   (b) Apply the initial condition(s) to find how the amount of chlorine changes in time during Stage 1.
   (c) Does your answer make sense physically? Does your solution have the properties you expect it to have?
   (d) Write down any thoughts you have about your work, including what you think went well, what you are unsure of, and what you'd like to have feedback on.
3. Model formulation – Stage 2
   (a) Formulate a differential equation modeling the amount of chlorine during Stage 2. Define any new constants, if necessary.

   (b) Write down appropriate initial condition(s) for Stage 2, in order to couple the problem to that of Stage 1.

4. Use the method of integrating factors to solve the differential equation modeling Stage 2. In particular:
   (a) Identify an appropriate integrating factor.
   (b) Obtain a general solution to the differential equation.
   (c) Apply the initial condition(s) to find how the amount of chlorine changes in time during Stage 2.
   (d) Does your answer make sense physically? Does your solution have the properties you expect it to have, in comparison to that of Stage 1?
   (e) How long does it take for the concentration of chlorine to reach the recommended minimum value of 1.3×10^{-5} ounces per gallon of water?
   (f) Write down any thoughts you have about your work, including what you think went well, what you are unsure of, and what you'd like to have feedback on.
5. Swap your work with another group. Read the other group's work carefully and provide feedback on their work, taking time to respond to their questions and anything they have requested feedback for.
6. Discuss the feedback you have received from the other group. Provide each other the opportunity to ask questions about the feedback you have given each other.
7. Are there any areas you could improve on? Take the opportunity to implement any changes in your work, based on the feedback you have received.

Criteria

Hand in your work addressing the above questions as a group, in the form of a written report. You will be graded based on mathematical correctness, accuracy, clarity, and grammar.

Individual Project - Part II

This part should be completed individually.

Consider the two mathematical models that you have derived with your group for Stage 1 and Stage 2.

1. Apply Euler's method to create a MATLAB script that implements a numerical algorithm to solve the two model equations numerically. The script should let the user specify the step size.
2. Write supporting documentation outlining what each part of your script does.

3. Using the exact solution that you have obtained with your group, plot the numerical error as a function of time using the step size $h = 0.1$ for $0 < t < 24 \times 60$ (minutes) for Stage 1 and $24 \times 60 < t < 48 \times 60$ (minutes) for Stage 2, in a single figure.
4. Plot the numerical error as a function of time using the step sizes $h = 0.02$ and $h = 0.01$ for both Stage 1 and Stage 2 in a single figure (overlain on top of the figure of Step 3 above).
5. Prepare a table listing the step size $h = 0.1, 0.09, \ldots 0.01$ in the first column and the maximal error over the time interval $0 < t < 48 \times 60$ (minutes) in the second column. What do you observe? *Hint: It may help to apply the command* `format short e`*, if the errors end up being small enough to unnecessarily truncate to zero.*

Criteria

Hand in your work for Part II of this project individually in the form of a report. You will be graded based on mathematical correctness, accuracy, clarity, and grammar.

REFERENCES

[1] Aprea, Maria-Cristina; Banchi, Bruno; Lunghini, Liana; Pagliantini, Massimo; Peruzzi, Antonio; Sciarra, Gianfranco. 2010. Disinfection of swimming pools with chlorine and derivatives: formation of organochlorinated and organobrominated compounds and exposure of pool personnel and swimmers. *Natural Science*. 2(2):68-78. https://www.scirp.org/pdf/NS20100200001_82150606.pdf.

[2] WHO. 2006. *Guidelines for safe recreational water environments. Volume 2: Swimming pools and similar environments*. Geneva SWITZERLAND: World Health Organization. https://www.who.int/water_sanitation_health/bathing/srwe2full.pdf