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STUDENT VERSION
Dimensionless Variables and Scaling for Differential Equations

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Abstract:

This material introduces the idea of "rescaling" for ordinary differential equations (ODE's) by the introduction of dimensionless variables. In practice this is an extremely common and useful prelude to the analysis and solution of ODE's, and yet is not often taught in an introductory course. Most ODE texts do not cover the material, but the benefits of understanding these techniques are myriad: rescaling indicates the appropriate physical scale on which the relevant independent and dependent variables exist, and can help determine conditions under which certain terms in an ODE model can be safely ignored, sometimes leading to great simplification. Proper scaling is also a useful prelude to the numerical solution of ODE's, and can improve the stability and efficiency of numerical methods.

MOTIVATION: A SPRING-MASS PROBLEM

Consider a classic spring-mass system: a mass of m kilograms is constrained to move along a line and is attached to a spring; refer to Figure 1, in which the mass moves only along the horizontal axis (no gravity!) Hooke's Law models the force F exerted by the spring on the mass as given by $F = -kx$, where x is the displacement of the mass from equilibrium ($x = 0$) and $k > 0$ is some constant that depends on the stiffness of the spring. Note that in order for $F = -kx$ to balance dimensionally k must have dimensions of force per distance (since $k = -F/x$). In metric MKS (meters-kilograms-seconds) units k has units of Newtons per meter or equivalently, kilograms per second squared.

Let us suppose that the spring is the only force acting on the mass, so there is no friction. From Newton's Second Law we have $F = ma$ where a is the acceleration of the mass. If the position of

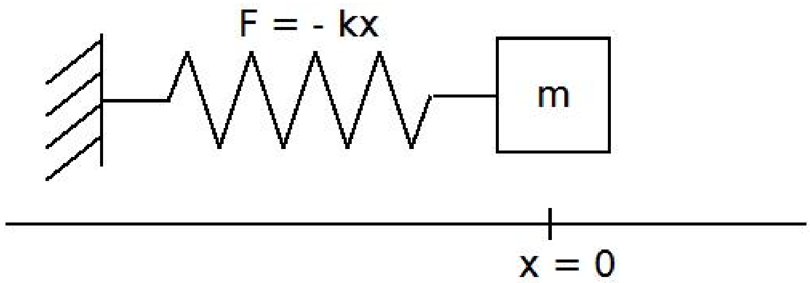

Figure 1. A spring-mass system.

the mass is given by $x(t)$ then $a = \frac{d^2x}{dt^2}$ and $F = ma$ leads to $-kx(t) = m\frac{d^2x}{dt^2}$ or

$$m\frac{d^2x}{dt^2} + kx(t) = 0, \tag{1}$$

an ordinary differential equation (ODE). Typical initial conditions are $x(0) = x_0$ (initial position) and $\frac{dx}{dt}(0) = v_0$ (initial velocity).

Exercise 1

Suppose the initial conditions are $x(0) = x_0$ for some x_0 and $\frac{dx}{dt}(0) = 0$. Show the solution to (1) is then $x(t) = x_0 \cos(t\sqrt{\frac{k}{m}})$, and that the period of the resulting motion is $P = 2\pi\sqrt{\frac{m}{k}}$. Verify that the quantity P so defined does indeed have the dimension of time.

CHARACTERISTIC VARIABLE SCALES

The period P of the spring-mass problem is an important feature of the physical system and is itself of interest, quite apart from the specific form of the solution. For example, if we want to measure the motion of an actual spring-mass system we should take data on a time scale comparable to P, perhaps 10 or 20 measurements per period, to clearly capture the motion of the system. Taking data more often than this might be overkill (if the period is 3 seconds it makes little sense to measure the mass position every microsecond.) Or perhaps we want to solve (1) numerically using Euler's method or something more sophisticated. The time scale of the problem will dictate the size of the time steps we take as we repeatedly extrapolate the solution forward in time. The time steps should be somewhat smaller than P if we are to track the motion accurately, but making them too small is wasteful.

In the spring-mass problem the period P is proportional to the quantity

$$t_c = \sqrt{\frac{m}{k}}. \tag{2}$$

Here t_c is the *characteristic time scale* of the problem. The quantity t_c, which does indeed have units of time, provides some indication of how fast the system is changing. For the spring-mass system we found this quantity by writing out the ODE (1), solving, and then finding P, but solving is a luxury we don't always have. However, it's usually possible to find the characteristic time scale for a problem without solving an ODE, or even writing down an ODE model! And indeed, more complex systems may have multiple time scales—different features of the physical system may change at different rates. Moreover, time isn't the only scale of interest. A system may have a characteristic length, or mass, or other physical dimension.

A BIT OF DIMENSIONAL ANALYSIS

Let's go back and find the time scale of the spring-mass system without solving or even writing down an ODE. First, we'll use "M" to denote the physical dimension of mass (not tied to any system units), "T" to denote the dimension of time, and "L" to denote length. The mass m in the spring-mass system has physical dimension M (duh!), which we'll indicate by writing $[m] = M$. From $F = ma$ and the fact that $[a] = LT^{-2}$ (acceleration has units of length per time squared) we find that $[F] = MLT^{-2}$. This means that the spring constant k has dimensions $[F/x] = MLT^{-2}L^{-1}$ or $[k] = MT^{-2}$.

The characteristic time scale of the spring-mass system ought to depend on the values of k and m, and in fact a simple hypothesis is that

$$t_c = k^a m^b \tag{3}$$

for some numbers a and b. Since $[t_c] = T$ we must choose a and b so that the right side of (3) has the dimension of time. Using $[m] = M$ and $[k] = MT^{-2}$, the right side of (3) has dimension

$$[k^a m^b] = (MT^{-2})^a M^b = M^{a+b}T^{-2a}.$$

Since both sides of (3) must have dimension T we need

$$\begin{aligned} a + b &= 0 \\ -2a &= 1. \end{aligned}$$

This yields $a = -1/2$ and $b = 1/2$. From (3) we conclude that $t_c = k^{-1/2}m^{1/2} = \sqrt{m/k}$. This is what we found in equation (2)!

Exercise 2

A pendulum of length R swings without friction in an environment with gravitational acceleration g. The mass of the bob at the end of the pendulum is m. Let t_c denote the characteristic time scale for the problem and suppose that

$$t_c = R^a g^b m^c$$

for constants a, b, c. Find the dimension of each of R, g, and m, and show that the characteristic time scale for this problem is given by $t_c = \sqrt{R/g}$ (and so the time scale does not depend on the mass of the bob, already a valuable piece of information!) Note that you don't actually need to use the ODE that governs a swinging pendulum.

Exercise 3

Consider a damped spring-mass system with mass m, spring constant k, and in which the frictional force on the mass is given by $-\mu x'(t)$ for some positive constant μ. The appropriate ODE is

$$m\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + kx(t) = 0 \tag{4}$$

with some initial conditions. Show that $[\mu] = MT^{-1}$ (use the fact that $\mu x'(t)$ has the dimension force). If

$$t_c = k^a m^b \mu^c$$

show that we need

$$a + b + c = 0$$
$$-2a - c = 1$$

and that any solution is of the form

$$a = b - 1, \quad c = 1 - 2b$$

where b can be chosen arbitrarily. As a result, any characteristic time scale must be of the form

$$t_c = k^{b-1}m^b\mu^{1-2b} = \frac{\mu}{k}\left(\frac{mk}{\mu^2}\right)^b \tag{5}$$

for some b. Note that the parenthesized expression mk/μ^2 is dimensionless. Equation (5) shows that there are infinitely many possible time scales!

Exercise 4

We can nail down a specific physically relevant time scale by choosing a value for b in (5); different choices highlight different time scales on which the phenomena evolves. Show that the choice $b = 1/2$ in (5) yields the previous time scale $t_c = \sqrt{m/k}$ that captures the period of the motion.

Exercise 5

How about a time scale that involves the damping constant μ? This could give insight into the rate at which the solution decays, rather than oscillates. It makes intuitive sense that as μ gets larger, the motion of the mass should damp to zero faster (though it never really gets there), so let's look for a time scale in which μ appears to a negative power. Show that the choice $b = 1$ in (5) yields another time scale $t'_c = m/\mu$.

To gain some insight into how t'_c relates to the solution, take $m = 1, \mu = 0.1, k = 1$ and solve the ODE (4) with $x(0) = 1, x'(0) = 0$ (analytically or numerically) and then plot the solution from $t = 0$ to $t = 200$; the solution should substantially decay to zero by around $t = 100$. Compare this to the value of m/μ. Then redo the computation but change the damping to $\mu = 0.2$. How long does it take the solution to decay to a comparable level now, and how does it relate to m/μ? Try $\mu = 0.5$ and $\mu = 1$, and compare m/μ to the decay time of the solutions. In plain English, what does the time scale m/μ quantify here?

Exercise 6

Consider the ODE (4) with initial conditions $x(0) = x_0$ and $\frac{dx}{dt}(0) = 0$. We now seek a characteristic length scale x_c for this problem, something of the form

$$x_c = k^a m^b \mu^c x_0^d.$$

Given that $[x_0] = L$, show that anything of the form

$$x_c = k^b m^b \mu^{-2b} x_0 = x_0 \left(\frac{mk}{\mu^2}\right)^b$$

has dimension length. What's the simplest choice for b?!

ANOTHER MOTIVATIONAL EXAMPLE

Consider a spring-mass system with a nonlinear "hard" spring that exerts a restoring force of the form $F(x) = -kx - qx^3$ for positive constants k and q. This models a spring that exerts more force than would be dictated by Hooke's Law, especially for large displacements. Suppose the mass is displaced by an amount x_0 at $t = 0$ and released with no initial velocity. The same $F = ma$ argument that led to (1) yields

$$m\frac{d^2x}{dt^2} + kx + qx^3 = 0, \tag{6}$$

$$x(0) = x_0,$$

$$\frac{dx}{dt}(0) = 0.$$

This ODE is nonlinear and can't be solved in any simple closed form. However, if the nonlinear qx^3 term wasn't present then this would be the spring-mass problem we've already solved. Can we safely ignore that term and use the linear system as a good approximation?

Exercise 7

To gain some insight into when the qx^3 term may or may not significantly influence the solution, consider the case where $m = 1$, $k = 1$, and $q = 0.01$ in (6), for a couple different initial displacements x_0.

(a) Use a computer package to solve the ODE (6) numerically for initial displacement $x_0 = 1$, from $t = 0$ to $t = 25$. Plot the solution. Solve and plot again without the qx^3 term, and compare the solutions. Did the nonlinear term make much difference?

(b) Repeat the last exercise with $x_0 = 5$.

(c) Repeat the procedure with $x_0 = 5$, but now take $k = 20$.

DIMENSIONLESS VARIABLES AND SCALING

In Exercise 7 you should find that, depending on the values of k and x_0, $q = 0.01$ might be small enough that the qx^3 terms makes little difference, or it might make a lot of difference! Maybe it depends on m too. Trying to sort out when a term like this is important and when it's not can be quite difficult. But a rescaling or "change-of-variable" argument can make it much easier analyze.

Let's start with a simple example of how rescaling works. In the ODE (1) with initial conditions $x(0) = x_0$, $\frac{dx}{dt}(0) = 0$, the independent variable t has physical dimension "time," of course, while the dependent variable x has dimension "length." The relevant time scale on which t evolves is given by (2) and from Exercise 6 above, the relevant spatial scale on which $x(t)$ changes is $x_c = x_0$. We will make a change of variables in both time and space to make both variables "dimensionless."

Specifically, define a new independent time variable $\bar{t} = t/t_c$ with t_c as given in (2), and also define a new dependent spatial variable $\bar{x} = x/x_c$ where $x_c = x_0$ (from Exercise 6). Both $\bar{t}$ and $\bar{x}$ are dimensionless—they are "pure numbers" with no units in whatever system of units we work in. We can move back and forth between the t and $\bar{t}$ time scales (since $t = t_c\bar{t}$), as well as the x and $\bar{x}$ systems ($x = x_c\bar{x}$).

From Exercise 1, the motion of the mass in the original (t, x) variables is given by $x(t) = x_0 \cos(t\sqrt{k/m})$, but the motion can also be described in the dimensionless $(\bar{t}, \bar{x})$ variables. We use a new function $\bar{x}(\bar{t})$ to indicate the position of the mass, so that for any choice of $\bar{t}$ (which corresponds to some time $t = t_c\bar{t}$ in the original time variable) we have

$$\bar{x}(\bar{t}) = \frac{x(t)}{x_c}. \tag{7}$$

Since $t = t_c\bar{t}$ we can also write this explicitly, $\bar{x}(\bar{t}) = \frac{x(t_c\bar{t})}{x_c}$. Alternatively, since $\bar{t} = t/t_c$ (7) also yields $x(t) = x_c\bar{x}(t/t_c)$.

The function $\bar{x}(\bar{t})$ satisfies an ODE similar to (1), but simpler. To see this, start with (7) in the form $x(t) = x_c\bar{x}(\bar{t})$ and differentiate both side with respect to t twice (use the chain rule, and note $d\bar{t}/dt = 1/t_c$) to find

$$\frac{dx}{dt} = x_c \frac{d\bar{x}}{d\bar{t}} \frac{d\bar{t}}{dt} = \frac{x_c}{t_c} \frac{d\bar{x}}{d\bar{t}} \tag{8}$$

$$\frac{d^2x}{dt^2} = \frac{x_c}{t_c} \frac{d^2\bar{x}}{d\bar{t}^2} \frac{d\bar{t}}{dt} = \frac{x_c}{t_c^2} \frac{d^2\bar{x}}{d\bar{t}^2} \tag{9}$$

Substituting these relations into the ODE (1) produces

$$\frac{mx_c}{t_c^2} \frac{d^2\bar{x}}{d\bar{t}^2} + kx_c\bar{x} = 0.$$

However, $t_c^2 = m/k$ and so the above equation collapses to

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} = 0. \tag{10}$$

after dividing by kx_c. The general solution to (10) is just $\bar{x}(\bar{t}) = c_1 \cos(\bar{t}) + c_2 \sin(\bar{t})$. The initial condition for $\bar{x}(0)$ is $\bar{x}(0) = x(0)/x_c = x_0/x_c = 1$! Thus we find that we must take $c_1 = 1, c_2 = 0$ in the general solution and so

$$\bar{x}(\bar{t}) = \cos(\bar{t}).$$

Of course, we can translate this back to $x(t)$ using $x(t) = x_c \bar{x}(t/t_c)$ and find

$$x(t) = x_c \cos(t/t_c) = x_0 \cos(t\sqrt{k/m}).$$

RESCALING FOR APPROXIMATION

But the point above is not simply to solve the ODE (1) by rescaling to a simpler ODE. When we properly rescale variables, it not only simplifies the ODE of interest, but also makes it much easier to see which terms are important and which might have a negligible effect on the solution (and so can be omitted), and how this depends on the various parameters in the ODE. Let's return to the hard spring system governed by (6) as an example.

Since our goal is determine when the qx^3 term might be negligible so that the solution to the ODE (1) suffices as an approximation, let's again rescale with dimensionless variables $\bar{t} = t/t_c$ and $\bar{x} = x/x_c$, where t_c is given by (2) and $x_c = x_0$. With $\bar{x}(\bar{t}) = x(t)/x_c$ as before, (8) and (9) still hold. Using these in (6) yields

$$\frac{mx_c}{t_c^2}\frac{d^2\bar{x}}{d\bar{t}^2} + kx_c\bar{x} + qx_c^3\bar{x}^3 = 0.$$

Use $t_c^2 = m/k$, $x_c = x_0$, and divide through by mx_c to find

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \bar{x} + \epsilon\bar{x}^3 = 0 \tag{11}$$

where

$$\epsilon = \frac{qx_0^2}{k}. \tag{12}$$

The initial conditions become $\bar{x}(0) = 1$ and $\frac{d\bar{x}}{d\bar{t}}(0) = 0$. You can easily check that ϵ is dimensionless. If we solve (11) for $\bar{x}(\bar{t})$ we can recover the solution $x(t)$ to (6). Moreover, if we take $\epsilon = 0$ we obtain the solution to (1).

This helps make it clear when the nonlinear term can be neglected. If we let $\bar{x}_\epsilon(\bar{t})$ denote the solution to (11) for a fixed ϵ (so $\bar{x}_0(\bar{t}) = \cos(\bar{t})$) we expect that $\bar{x}_\epsilon$ is well-approximated by $\bar{x}_0$ when ϵ is sufficiently close to 0. To illustrate, in the left panel of Figure 2 is shown $\bar{x}_\epsilon(\bar{t})$ versus $\bar{x}_0(\bar{t}) = \cos(\bar{t})$ when $\epsilon = 0.05$, for 5 periods, so $0 \leq \bar{t} \leq 10\pi$. The same information is shown in the right panel for $\epsilon = 1$. If we consider the approximation on the left sufficient for our purposes (but not on the right) then something like $\epsilon \leq 0.05$ makes sense as a criteria for dropping the nonlinear term. But in light of (12) this is just the condition

$$\frac{qx_0^2}{k} \leq 0.05. \tag{13}$$

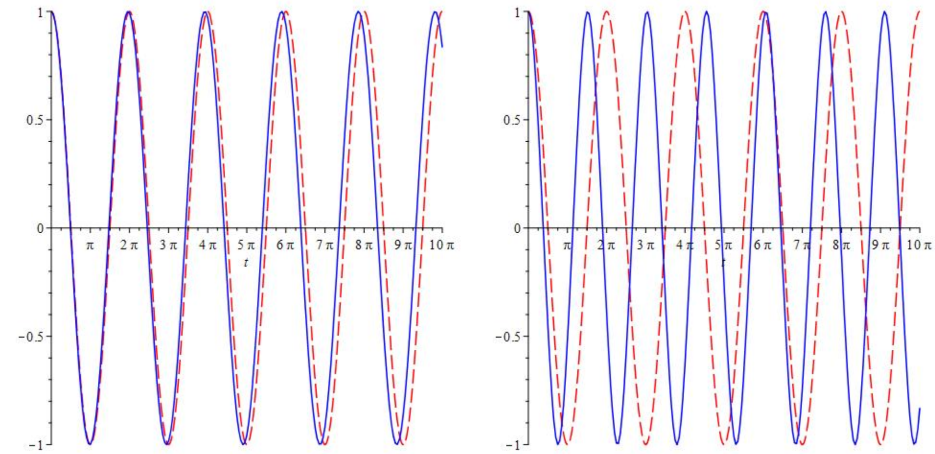

Figure 2. Solution to (11) (solid curve) versus cos(t) (dashed curve) when $\epsilon = 0.05$ (left) and $\epsilon = 1$ (right).

If (13) holds we are justified in dropping the nonlinear term in (6), at least if we're only interested in the first five periods of the solution. If we want the solutions to agree for a longer time period, or agree even more closely, we might need to lower 0.05. In any case, the critical quantity that dictates when the solution to the nonlinear problem (6) agrees with the solution to the linear problem (1) is the dimensionless quantity qx_0^2/k. This also shows that the mass m plays no role in when the approximation is valid.

SUMMARY

Rescaling via the introduction of dimensionless variables can make the analysis and solution of ODE's (and other problems in applied mathematics!) much easier. In particular

- It can illuminate the nature of the solution to an ODE by indicating the natural scale for the quantities of interest, in time, space or other physical dimensions.
- It allows us to determine when and how equations can be simplified by dropping "negligible" terms.
- It can aid in the numerical solution of the problem. For example, if we want to solve (6) numerically for a large selection of different values of m, k, q, and x_0, the introduction of dimensionless variables shows that we really only need to solve (11) for the relevant values of ϵ, which may lead to much less work.
- As it turns out, the introduction of dimensionless variables is an essential part of many types of experimentation. For example, you can't easily put a full scale jumbo jet into a wind tunnel; you'd likely use a scale model. But will the results for the scale model correspond to the behavior of a full size jet? Dimensional analysis of the type we've done can help assure that it will. See [1] for more on this topic and dimensional analysis in general.

Exercise 8

Consider the damped spring-mass equation (4) with initial conditions $x(0) = x_0$, $\frac{dx}{dt}(0) = 0$. Rescale the problem to the dimensionless form

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \epsilon\frac{d\bar{x}}{d\bar{t}} + \bar{x} = 0$$

with appropriate initial conditions using $x_c = x_0$ and $t_c = \sqrt{m/k}$.

What is ϵ in terms of m, μ, k, and x_0? Compare the damped and undamped solutions for a few values of ϵ, say over the first five cycles. Under what conditions (analogous to (13)) on m, μ, k, and x_0 can we reasonably ignore the damping for five cycles?

Exercise 9

A series RC (resistor-capacitor) circuit is governed by the ODE

$$R\frac{dq}{dt} + \frac{q(t)}{C} = 0 \tag{14}$$

where R is the resistance of the resistor, C is the capacitance of the capacitor, and $q(t)$ is the charge on the capacitor. We assume there is no voltage source in the circuit. Suppose the capacitor starts with initial charge $q(0) = q_0$.

Let us use the symbol "Q" for the fundamental physical dimension of electric charge, so for example, $[q] = Q$. It turns out that voltage or electrical potential is "work per unit charge." If we take work as "force times distance" then work has dimension ML^2T^{-2}, and so voltage V has dimension $[V] = ML^2T^{-2}Q^{-1}$. The fundamental relation that governs a capacitor is $q = CV$, and so $[C] = Q/[V] = M^{-1}L^{-2}T^2Q^2$. Similarly from Ohm's law $V = IR$ (where I is current in the circuit, so $[I] = QT^{-1}$, charge per time) we find $[R] = [V/I] = ML^2T^{-1}Q^{-2}$.

(a) Use $[R] = ML^2T^{-1}Q^{-2}$ and $[C] = M^{-1}L^{-2}T^2Q^2$, along with $[q_0] = Q$ to find a time scale t_c for this ODE, of the form

$$t_c = R^aC^bq_0^c.$$

(b) Find a "charge scale" q_c for the problem, also in the form $q_c = R^aC^bq_0^c$ (different a, b, c than for t_c.)

(c) Let $\bar{t} = t/t_c$ and $\bar{q} = q/q_c$. Show that the rescaled ODE involving $\bar{q}(\bar{t})$ is

$$\frac{d\bar{q}}{d\bar{t}} + \bar{q} = 0$$

with $\bar{q}(0) = 1$. Solve this DE and use it to find the solution to (14).

Exercise 10

Suppose an object of mass m falls at velocity $v(t)$ under the influence of gravitational acceleration g. Let's take $v > 0$ to indicate downward motion, which is all we're interested in. As it falls the mass experiences a drag force $F(v)$ that is a function of the object's velocity v. Suppose that

$F(v) = -k_1 v - k_2 v^2$ for some positive constants k_1 and k_2, so that the force is always upward (which is the negative direction) when $v > 0$. From $F = ma$ we find

$$m\frac{dv}{dt} = g - k_1 v(t) - k_2 v^2(t). \tag{15}$$

Note $g > 0$ here.

(a) What are the dimensions of k_1 and k_2? (Hint: $F(v)$ is a force, so $k_1 v$ and $k_2 v^2$ must be forces as well).

(b) Show that any characteristic time that can be formed from m, g, k_1, and k_2 is of the form

$$t_c = \frac{m}{k_1}\left(\frac{mgk_2}{k_1^2}\right)^c$$

for some choice of c (note the quantity in parentheses is dimensionless). Hint: mimic the procedure for t_c after equation (4) in Exercise 3.

(c) Show that any characteristic velocity that can be formed from m, g, k_1, and k_2 is of the form

$$v_c = \frac{mg}{k_1}\left(\frac{mgk_2}{k_1^2}\right)^d$$

for some choice of d.

(d) Let $\bar{v}(\bar{t}) = v(t)$ where $\bar{v} = v/v_c$ and $\bar{t} = t/t_c$, with t_c and v_c are obtained from the above formulae, both with $d = 0$. Show that this leads to an ODE for $\bar{v}$ of the form

$$\frac{d\bar{v}}{d\bar{t}} = 1 - \bar{v} - \epsilon\bar{v}^2 \tag{16}$$

where $\epsilon = mgk_2/k_1^2$ (dimensionless).

Solve (16) with $\epsilon = 0$ and $\bar{v}(0) = 0$. Call this solution $\bar{v}_0(t)$. Then compute (numerically or symbolically) the solution to (16) with $\bar{v}(0) = 0$ for each of $\epsilon = 0.01, 0.1, 1.0$ and plot with $\bar{v}_0(t)$ for $0 \le t \le 5$. Experiment. How small must ϵ be before the solution to (16) agrees well with $\bar{v}_0$ on this time interval?

Based on this, what is a reasonable quantitative criteria (involving m, g, k_1, and k_2) for dropping the quadratic term in $F(v)$ and using the simpler linear ODE?

(e) Find a choice for c and d in the formulae for t_c and v_c above that leads to values of t_c and v_c so that the rescaled ODE is of the form

$$\frac{d\bar{v}}{d\bar{t}} = 1 - \epsilon\bar{v} - \bar{v}^2. \tag{17}$$

Mimic the computations in part (d) above to find a reasonable quantitative criteria (involving m, g, k_1, and k_2) for dropping the linear term in $F(v)$ and using the ODE in which $F(v)$ is purely quadratic.

REFERENCES

[1] Barenblatt, G.I., 2003. *Scaling*. New York: Cambridge Texts in Applied Mathematics (Volume 34).