

**STUDENT VERSION**

**CHEBYSHEV POYNOMIAL SOLUTION**

Gabriel B. Costa

Department of Mathematical Sciences

United States Military Academy

West Point NY 10996 USA

gabriel.costa@westpoint.edu

**STATEMENT**

**Introduction:** Differential equations are considered *classical* if they have been studied for many years and their solutions have a variety of applications. These equations appeal to the pure mathematician because of their aesthetic beauty and to the scientist and engineer for their applicability.

One such example is the ***Chebyshev*** differential equation, which is given by:



This equation possesses **finite** polynomial solutions (as opposed to infinite series solutions), and we call these solutions the *Chebyshev* *Polynomials.*

**The series solution technique:** way to obtain finite polynomial solutions to the aforementioned differential equation is by using the *series method* studied in class. That is, since  is an *ordinary point* for this differential equation, i.e. the lead polynomial, $P\left(x\right)=1- x^{2}$, is not 0 when , we can assume a solution of the form and, after differentiating term-by-term, we substitute and into the differential equation. After completing the appropriate manipulations, we collect like terms, compare coefficients of powers of *x*, and eventually obtain a *recursion formula*. We then “split off” the two linearly independent solutions which are guaranteed by existence theorems. We recall that both  and  are arbitrary.

**The method verified:** Let in the Chebyshev differential equation. Obtain two solutions to the differential equation via series techniques.

* One finite polynomial solution exists. What is the degree of this polynomial?
* What observations can be made about the other solution?
* What values of  and  yield only a polynomial solution and no “infinite solution”?
* What happens if and? Make a plausible conjecture as to the degrees of the various finite polynomial solutions and the values imposed on  and  to yield only polynomial solutions.

**Historical background:** Investigate the history of this equation. Who was Chebyshev? What properties do the polynomial solutions enjoy? What are some of the concepts associated with this differential equation? What are some of the applications which use this equation and the associated polynomial solutions?

**A similar differential equation with two polynomial solutions:** Consider thedifferential equation below, where and *B* are integers which can regarded as parameters:



Note that this equation is similar to the Chebyshev differential equation.

* Can we find conditions on the parameters *A* and *B* such that the differential equation will possess *two* polynomial solutions?
* Furthermore, can we determine the degrees of the polynomial solutions from the values of *A* and *B*? Can we answer these questions by merely *inspecting* the differential equation?

Formalize the results into a **Theorem**, supported by a **Proof**.