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# STUDENT VERSION
# CONJECTURING SOLUTIONS FOR LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

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**Abstract:** We lead students from the solution for $y(t) = k\,y(t)$ to a natural extension for the solution conjecture for a system of two constant coefficient, homogeneous, linear differential equations introducing eigenvalues and eigenvectors through student discovery.

## INTRODUCTION

We consider a system of linear differential equations

$$\begin{aligned} x'(t) &= a\,x(t) + b\,y(t) \\ y'(t) &= c\,x(t) + d\,y(t) \end{aligned} \tag{1}$$

with $x(0) = x_0$ and $y(0) = y_0$.

We can put this in matrix form:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a\,x(t) + b\,y(t) \\ c\,x(t) + d\,y(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{2}$$

with $x(0) = x_0$ and $y(0) = y_0$.

OR

$$X'(t) = A \cdot X(t)\,,$$

where

$$X'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \quad , \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad , \quad X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{3}$$

$X'(t) = A \cdot X(t)$ is very suggestive for a solution of the form $X(t) = ce^{At}$. Thus one could conjecture a solution and try it out to see what $c$ would have to be, but we have to remember $A$ is a matrix, not a number, and while there are theories of what $e^A$ would mean for a matrix $A$ we choose not to go there now.

Instead, we conjecture a vector solution for $X(t)$ which has the best of both worlds (1) nice exponential form with no matrices in the exponent and (2) vector status where we can see the pieces of the solution:

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \cdot e^{\lambda t}. \tag{4}$$

So here we are suggesting $x(t) = ue^{\lambda t}$ and $y(t) = ve^{\lambda t}$.

**Assignment**

1. Now put the conjecture (4) in (3) and see what has to happen to $a$, $b$, $c$, $d$ and $u$, $v$ in order to obtain solutions - interesting solutions, i.e. other than $u = 0$ and $v = 0$. Go for it!!! Summarize your conclusions in a nice write-up
2. Illustrate the solution strategy from assignment (1) using the following system

$$\begin{aligned} x'(t) &= -4\,x(t) - 2\,y(t) \\ y'(t) &= -1\,x(t) - 3\,y(t) \end{aligned} \tag{5}$$

with $x(0) = 2$ and $y(0) = 1$.