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STUDENT VERSION
Skin Burn Model - Numerical Methods

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Abstract: The heat equation is an important partial differential equation (PDE) which describes the distribution of heat in a given region over time. Here we learn to solve a heat equation numerically. It is difficult to study the behavior of temperature in problems with interfaces analytically so numerical methods play an important role in solving these. The idea presented here can be used to solve a wide variety of models for non-homogenous inner structure models with partial differential equations.

SCENARIO DESCRIPTION - BACKGROUND READING

The heat equation arises in various fields such as image processing, hyperthermia treatment, chemical reactions, etc.and it can be applied to the study of skin burns. The skin is the largest organ of the human body and plays an important role such as defense, sensory effects, and thermoregulation. Skin burns are one of the most devastating injuries encountered in a human's daily life. These injuries usually result from heat, electricity, sunburn, radiation or chemicals. Skin tissue has a complicated multilayer structure namely epidermis, dermis and subcutaneous (see Figure 1) and the complex boundary and interfacial conditions raise the difficulty of solving the problem analytically. Each layer possesses different thermo-physical properties. Using Numerical Methods we develop algorithms which enable a computer to find the solution of complicated problems.

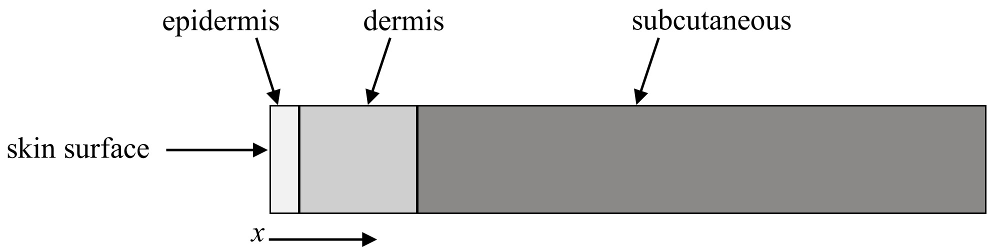

Figure 1. Triple layered skin structure

HEAT ENERGY MODEL

In any material the thermal energy transfers from the region of higher temperature to the region of lower temperature. Consider a uniform rod of length l with density ρ and specific heat c with cross-sectional area of the rod to be A. κ is called the thermal conductivity.

Let $u(x,t)$ be the temperature of the thin slice of the rod between x and $x+\Delta x$. We offer a derivation of the heat equation which is based on Fourier's law and the Conservation of Energy Principle. Basically, we will compute the change in heat in a small section of the rod in two different manners and equate these to form a partial differential equation.

Now, the heat energy of the segment between x and $x+\Delta x$ is computed by

$$c\rho A\Delta x u(x,t) \tag{1}$$

while Fourier's law of heat transfer gives the following:

$$\frac{\text{Rate of heat transfer}}{\text{area}} = -\kappa\Delta u(x,t)\,. \tag{2}$$

Using the Conservation of Energy Principle,

$$\text{change of heat energy in the segment in time } \Delta t\ = \text{energy in} - \text{energy out}\,, \tag{3}$$

from (1)-(3), we obtain a difference equation which is then simplified with division of both sides by the factor $A\Delta t\Delta x$.

$$c\rho A\Delta x(u(x,t+\Delta t)-u(x,t)) = \Delta t A(-\kappa\Delta u(x,t)+\kappa\Delta u(x+\Delta x,t)) \tag{4}$$

$$c\rho\frac{(u(x,t+\Delta t)-u(x,t))}{\Delta t} = \frac{(-\kappa\Delta u(x,t)+\kappa\Delta u(x+\Delta x,t))}{\Delta x} \tag{5}$$

Taking the limit as Δt and Δx both tend to 0 gives the Heat Equation in our case, a partial differential equation:

$$(\kappa u_x)_x = \rho c\frac{\partial u(x,t)}{\partial t} \tag{6}$$

SKIN BURN MODEL

To predict the behavior of elevated temperature in skin burn treatment, a model is as follows (see [1, 2, 3, 4, 5]):

$$(\kappa u_x)_x - w_b c_b u(x,t) + f = \rho c \frac{\partial u(x,t)}{\partial t} \tag{7}$$

where ρ and c denote density and specific heat of the tissue; c_b is the specific heat of the blood; w_b blood perfusion rate; f is internal heat source. (f is the heat generated due to metabolism)

NUMERICAL APPROACH TO SOLVE THE MODEL

First we discretize the region $\{(x,t) : a \leq x \leq b, t \geq 0\}$ into N subintervals in space direction with spacing of $h = \frac{(b-a)}{N} > 0$, so that $x_{l+1} - x_l = h$. Let Δt be the time step. For $l = 0, 1, 2, ..., N$ and $j > 0$, we let the solution u at the grid point (x_l, t_j) be denoted by u_l^j. We can now approximate the time derivative using Forward Euler's method with,

$$u_{t_l}^j = \frac{u_l^{j+1} - u_l^j}{\Delta t} \tag{8}$$

and spatial derivative u_{xx} by central differencing

$$u_{xx_l}^j = \frac{u_{l+1}^j - 2u_l^j + u_{l-1}^j}{h^2}\,. \tag{9}$$

Substituting (8) and (9) in PDE (7),

$$\rho c \frac{u_l^{j+1} - u_l^j}{\Delta t} = \kappa \frac{u_{l+1}^j - 2u_l^j + u_{l-1}^j}{h^2} - w_b c_b u_l^j + f_l^j \tag{10}$$

we obtain a system of algebraic equations:

$$u_l^{j+1} = u_l^j + \frac{\kappa \Delta t}{\rho c} \frac{u_{l+1}^j - 2u_l^j + u_{l-1}^j}{h^2} - \frac{w_b c_b \Delta t}{\rho c} u_l^j + \frac{\Delta t}{\rho c} f_l^j\,. \tag{11}$$

Suppose we have initial temperature as

$$u(x,0) = T_0 \tag{12}$$

$$u_l^0 = T_0 \text{ for all } l \tag{13}$$

and temperature at the boundaries as

$$u(a,t) = T_1(t), u(b,t) = T_2(t) \tag{14}$$

$$u_o^j = T_1(t_j), u_N^j = T_2(t_j) \text{ for all } j \tag{15}$$

Initial Actions

For numerical experiments we take thickness of the tissue as 12.08 mm, density $\rho = 0.001 g/mm^3$, specific heat $c = 4.2 J/g^oC$, $c_b = 4.2 J/g^oC$, perfusion rate $w_b = 5 * 10^{-7} g/mm^3/sec$, thermal conductivity $\kappa = 5.2 * 10^{-4}$ and metabolic heat f=0

1. Actions 1: When skin comes in contact with hot steel plate, skin surface temperature is constant. Taking elevated temperature 12^{o}C on the skin boundary, solve the system (11) using MATLAB or any other software.
2. Actions 2: Plot the graph of temperature versus depth at time 30 sec as shown in Figure 2.
3. Actions 3: Observe how temperature behaves at different depths. Figure 3 shows the behavior of temperature at depths 0.72mm and 2.08mm as time progresses.

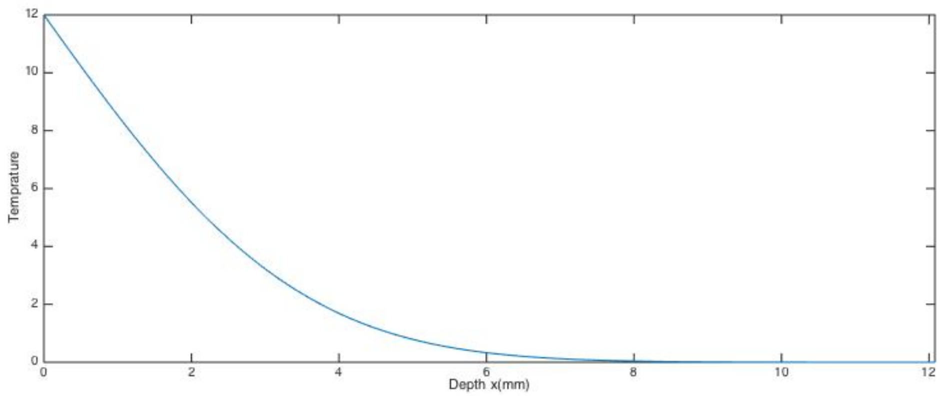

Figure 2. Temperature versus depth graph

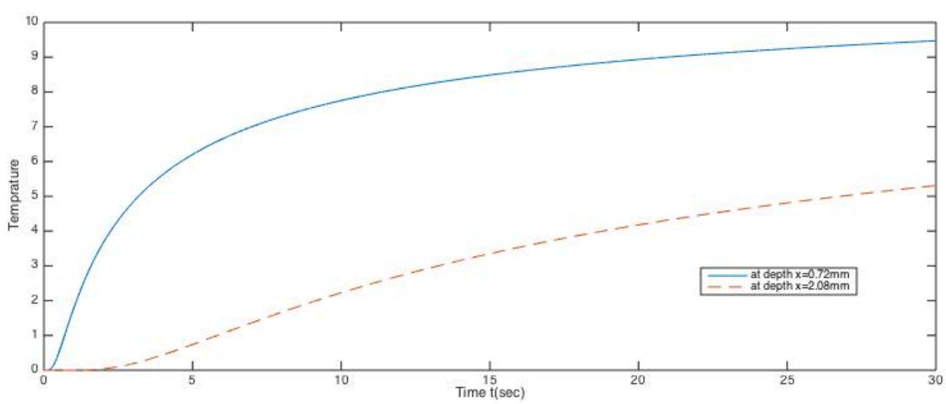

Figure 3. Time versus temperature graph at two different depths

FURTHER MODIFICATIONS IN THE MODEL

There are different layers of skin with different physical and thermal properties. Suppose two layers meet at grid point $x = x_{l_1}$. The interface separates the region into two parts $\Omega^- = (a, x_{l_1})$ and $\Omega^+ = (x_{l_1}, b)$. Let κ, ε, and σ be piecewise constant functions, i.e.

$$\kappa = \begin{cases} \kappa^- & \text{in } \Omega^- \\ \kappa^+ & \text{in } \Omega^+ \end{cases}, \quad \varepsilon = \begin{cases} \varepsilon^- & \text{in } \Omega^- \\ \varepsilon^+ & \text{in } \Omega^+ \end{cases}, \quad \sigma = \begin{cases} \sigma^- & \text{in } \Omega^- \\ \sigma^+ & \text{in } \Omega^+ . \end{cases}$$

Physically the temperature should be continuous at the interface also which gives

$$u^+ - u^- = \lim_{x \to \ l_1{}^+} u(x) - \lim_{x \to \ l_1{}^-} u(x) = 0 \,. \tag{16}$$

In addition, continuity of heat flux is applied to the boundary of adjacent layers,

$$\kappa^+ u_x^+ = \kappa^- u_x^- \,. \tag{17}$$

Now using Taylor's expansion about x_l, we have

$$u_{l_1-1} = u^- - u_x^- h + \frac{1}{2} u_{xx}^- h^2 + O(h^3) \quad \text{and} \tag{18}$$

$$u_{l_1+1} = u^+ + u_x^+ h + \frac{1}{2} u_{xx}^+ h^2 + O(h^3) \,. \tag{19}$$

Solving equations (7), (16)-(19) for u_x^- and u_x^+, we obtain

$$\begin{aligned} (\rho^+ c^+ + \rho^- c^-) u_{l_1}^{j+1} &= (\rho^+ c^+ + \rho^- c^-) u_{l_1}^{j} + 2\Delta t \frac{\kappa^+ u_{l_1+1}^{j} - (\kappa^+ + \kappa^-) u_{l_1}^{j} + \kappa^- u_{l_1-1}^{j}}{h^2} \\ &\quad - \Delta t (w_b^- c_b^- + w_b^+ c_b^+) u_{l_1}^{j} + \Delta t (f^+ + f^-) \end{aligned} \tag{20}$$

Activities

1. Solve the model numerically using MATLAB or any other software and predict the behavior of temperature at different layers when constant temperature 12^oC is given to the skin boundary. The values of the parameters is given in Table 1.

	Blood	Epidermis	Dermis	Subcutaneous
density [g/mm^3]		0.0012	0.0012	0.001
specific heat [J/g^{o}C]	4.2	3.6	3.4	3.06
thermal conductivity [w/mm^{o}C]		$2.6 * 10^{-4}$	$5.2 * 10^{-4}$	$2.1 * 10^{-4}$
perfusion rate [g/mm^3/sec]		0	$5 * 10^{-7}$	$5 * 10^{-7}$
thickness [mm]		0.08	2	10

Table 1. Values of the parameters used for the numerical experiments

2. Plot the graph of temperature versus depth.

3. Observe how temperature behaves at different depths.
4. When skin gets burned one can get relief by putting the skin under the cold water tap or under any cold beverage available at the moment. What conditions need to be imposed so that the model predicts the effect of cool pack on the each layer.
5. Examine the case when thermal conductivity is variable.
6. Predict the behavior of temperature at different depths when linear temperature is imposed on the skin boundary.
7. Apply this technique on other non-homogeneous models.

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