**Curvature and Maximum Permissible Speeds**

Note: Be sure to download the data file `track.tsv` before beginning this activity.

## Introduction

In class we've defined the concept of the **curvature** ($\kappa$) of a curve at a point, which roughly measures how "sharply" a curve turns at that point. Curvature is important in certain design problems where we want to prevent the direction of a moving object from changing too quickly. One common example of this from railway engineering is the problem of setting **maximum permissible speeds** for trains on sections of track. If a train rounds a corner too quickly there is a danger that it might tip or derail, so in general sharper turns (i.e. turns with greater curvature) require lower maximum permissible speeds.

Suppose we're trying to set the maximum permissible speed for a section of a commuter railway, and that our models show that the maximum safe speed $v_{\max}$ (in meters/second) for a train at a particular point is given by

$$v_{\max} = 2.5\sqrt{1 + \frac{1}{\kappa}} \tag{1}$$

where $\kappa$ is the curvature of the track (in meters$^{-1}$) at that point. Notice that $v_{\max}$ decreases as $\kappa$ increases, since sharper turns require lower speed. The track in question is level, and so has a constant $z$-coordinate of zero, meaning that we can describe its shape in $\mathbb{R}^3$ using a vector-valued function $\vec{r}$ of the form

$$\vec{r}(t) = \langle x(t), y(t), 0 \rangle \tag{2}$$

However we don't have access to a closed-form mathematical definition for the component functions $x$ and $y$ of $\vec{r}$. Instead the track is defined using a large data table of discrete coordinates gathered from survey measurements (this is very common in real-world applications; it's rare for us to have a clean, simple function that describes something, and far more common for us to have to rely on large sets of discrete points). The shape of the track is shown in Figure 1.

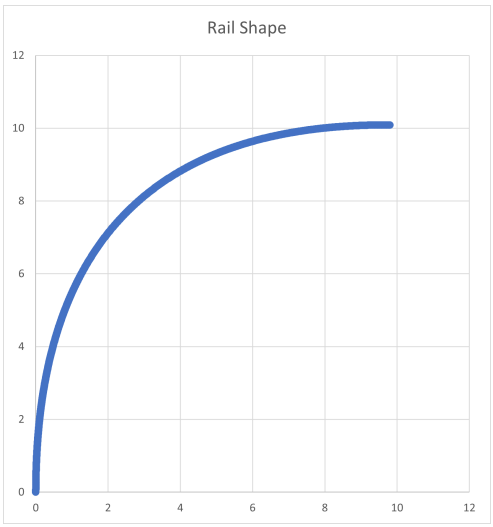


Figure 1: The shape of the train track defined in `track.tsv` (distances in meters). The points displayed in this graph are the $(x, y)$-pairs from the rows of the data file. The track looks roughly like part of a circle of radius 10 m, but is actually a more complicated shape that changes curvature across its length.

### Objective

Your goal is to find the maximum permissible speed for the section of track defined in `track.tsv`. This will require you to figure out how to numerically approximate the curvature of a vector-valued function defined as a data set, to find the maximum curvature on the track, and then to apply that to the model in equation (1) to find the maximum speed permitted by that curvature.

## Data and Software

The shape of the train track is defined in the included data file `track.tsv`. It is a plain text file whose rows each contain a parameter value $t$, an $x$-coordinate $x(t)$, and a $y$-coordinate $y(t)$, respectively, separated by tab characters ("`tsv`" stands for "tab-separated values"). All distances are in meters. You can open a `.tsv` file in any text editor to inspect the data table. Any spreadsheet program, or any file-reading method from a programming language, should be able to load a `.tsv` file in the form of a table, list, array, or equivalent data structure.

The data file is extremely long (approximately 1000 rows), so you will definitely need to process it with the aid of a computer. You are free to use any program you are comfortable with. If you are not familiar with programming, this entire project can be completed in a spreadsheet program like Microsoft Excel or a free alternative (e.g. Google Sheets[1] or LibreOffice Calc[2]). The only commands you'll need to know are how to add, subtract, multiply, and divide cells, how to evaluate square roots, absolute values, and exponents, and how to copy cell formulas to fill an entire column. Programming languages (e.g. Python or R) and mathematical software packages (e.g. MATLAB) are also options. Whatever you decide to use, you will need to know how to load the data in `track.tsv` and perform computations with the data in each row.

[1] `https://www.google.com/sheets/`

[2] `https://www.libreoffice.org/discover/calc/`

## Activities

Complete the following activities to find the maximum permissible speed on the track.

1. Finding $v_{\max}$ will require us to first find the curvature function $\kappa(t)$ of the track. If we had a closed-form definition for its shape function $\vec{r}(t)$, then we could just find its first and second derivatives and plug them into the curvature formula

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} \tag{3}$$

   However, our function $\vec{r}$ is defined as a table of discrete coordinates, so we cannot compute $\vec{r}\,'$ and $\vec{r}\,''$ exactly: we will instead have to approximate them using *average rates of change* between each row.

   In your program of choice, use the data in `track.tsv` to generate numerical approximations for $x'$, $y'$, $x''$, and $y''$ in each row[3].

2. Recalling that $\vec{r}\,'(t) = \langle x'(t), y'(t), 0\rangle$ and $\vec{r}\,''(t) = \langle x''(t), y''(t), 0\rangle$, we now have numerical approximations for both $\vec{r}\,'$ and $\vec{r}\,''$. Use them to approximate the value of $\vec{r}\,'(t) \times \vec{r}\,''(t)$ in each row.

3. Use these results to approximate $\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|$ and $\|\vec{r}\,'(t)\|^3$ in each row, and then use the curvature formula (3) to approximate the curvature $\kappa(t)$ in each row.

4. Find the maximum curvature on the entire track, and use it in the model equation (1) to determine the maximum permissible speed $v_{\max}$ (m/s).

[3]Depending on your chosen approximation method you might be unable to fill in the ends of the data series (e.g. using forward differences leaves no way to fill in the final entry since it has no "next" entry). That's not a problem for our purposes, so leaving the first or last few entries blank as necessary is fine.