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Hands-On Modeling with Pull-Back Cars

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SCENARIO DESCRIPTION

A pull-back car is a toy car powered by a clockwork motor [1]. Energy is stored in a flat spiral spring that winds up as the car is pulled backwards. When the car is released, the spring begins to unwind and the car accelerates forward. An ingenious design allows the internal spring to be wound up over a short distance, while providing locomotive force over a substantially longer distance as the car rolls forward.

Figure 1. Darda Race Car, c. 2015.

This modeling scenario applies a simple mathematical model to the prediction of velocity and distance traveled for a pull-back car. Students will find analytical solutions of the equation of motion in two different regimes in order to build a mathematical model of the car's behavior. Experimental data will be used to estimate the unknown constants in the model, which can then be used to predict the maximum velocity and distance traveled as a function of the pull-back distance.

Materials

Sample data and videos obtained from experiments with pull-back toys are provided. However, students are encouraged to conduct their own experiments throughout the project. To do so, students are advised to have the following materials on hand:

- A pull-back car;

- A digital video camera (e.g., a cellphone camera);
- A measuring stick or tape measure;
- A felt-tipped marker;
- A roll of masking tape.

1 The Mathematical Model

The equation of motion for a pull-back car can be simplified to the form

$$x''(t) + kx'(t) = M(x), \tag{1}$$

where $x(t)$ represents the position of the car at time t, k is a constant quantifying frictional losses, and $M(x)$ quantifies the torque (or moment) provided by the clockwork motor as a function of its position. The equation of motion is derived using Newton's Second Law and interested readers can find the details in another modeling scenario devoted to pull-back toys [2]. Notice that $M(x)$ must depend explicitly on the distance traveled, rather than time, because the internal spring winds down over a fixed distance rather than a fixed amount of time.

Pull-back cars can be wound up over a relatively short distance, yet the internal spring provides locomotive force over a much longer distance. This interesting design aspect of pull-back cars stems from separate gear trains that control the forward and backward motion of the toy. More precisely, let x_p denote the *pull-back* distance of the car, i.e., the distance the car is pulled backwards to wind up the internal spring. Similarly, let x_w represent the corresponding *wind-down* distance of the car, i.e., the distance the car will roll forward before the internal spring completely unwinds. The specific gear ratios used in the pull-back motor force x_w to be a constant multiple of x_p, which can be measured through a physical experiment.

It is natural to expect the function $M(x)$ to change as the pull-back distance, x_p, is changed. This modeling scenario adopts the *constant-torque* model for $M(x)$ developed in [2], which assumes that the spring within the clockwork motor provides constant torque as it unwinds, i.e.,

$$M(x) = \begin{cases} M_p, & 0 \le x \le x_w, \\ 0, & x > x_w, \end{cases} \tag{2}$$

where M_p is a quantity that depends on the pull-back distance, x_p. The relationship between x_p and M_p will be explored in the final section of this modeling scenario. The graph of $M(x)$ for this idealized model is shown in Figure 2. In order to account for changes in the frictional losses of the car after the motor has fully unwound, it is sensible to replace k in (1) by k_1 and k_2, which are used on the intervals $0 \le x \le x_w$ and $x \ge x_w$, respectively.

Student Task: Solve the mathematical model for the motion of the pull-back car by completing the series of exercises below.

Questions:

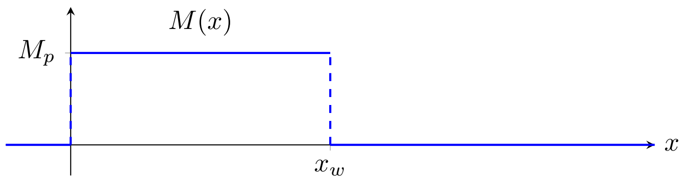


Figure 2. The form of the moment function for the constant-torque model.

1. Solve the equation of motion (1) over the range $0 \le x \le x_w$, where it reduces to

$$x''(t) + k_1 x'(t) = M_p$$

in light of (2). Assume that the car is initially at rest, i.e., $x(0) = x'(0) = 0$.

2. Solve the equation of motion (1) for $x \ge x_w$, where it reduces to

$$x''(t) + k_2 x'(t) = 0$$

in light of (2). What initial conditions should be considered? At what time should these initial conditions be imposed? Explain.

3. Sketch the graphs of $x(t)$ and $x'(t)$ as functions of time for the range $0 \le t \le 10$. Assume that $M_p = 650$, $k_1 = 2.0$, $k_2 = 0.5$, and $x_w = 150$. How far will the pull-back car travel as $t \to \infty$? What is its maximum velocity?

4. Using your solution from Question 3, graph the velocity $x'(t)$ as a function of $x(t)$. For what value of x does this graph reach its maximum? Does your answer make sense from an intuitive point of view? Why or why not?

2 Parameter Estimation

Section 1 focused on a simple mathematical model for the motion of a pull-back car. This portion of the project seeks to validate the model using experimental data that will allow for parameter estimation and independent observations of the distance traveled and maximum velocity. After choosing a specific pull-back distance, x_p, four model parameters must be determined from physical experiments with the pull-back car: x_w, M_p, k_1, and k_2. The wind-down distance x_w can be measured independently of the remaining constants. The constant M_p quantifies the torque provided by the pull-back motor. The constants k_1 and k_2, respectively, represent the decay rate of the car's speed due to frictional losses before and after reaching the wind-down distance. The two separate constants are needed to account for the fact that some components of the pull-back motor disengage once the spiral spring has wound down, producing a change in the frictional losses of the car. The three constants M_p, k_1, and k_2 must be determined together from experimental data observing the car in motion.

2.1 Estimation of x_w

One key aspect of the pull-back motor is that it can be wound up by pulling the car backwards over a short distance, yet the motor supplies torque to the wheels over a much longer distance as the car rolls forward. In the following task, students will choose a specific pull-back distance that should be used throughout Section 2. Students who do not have access to a pull-back toy can skip this section entirely and proceed to Section 2.2.

Student Task: Use the procedure below to estimate x_w and then answer the subsequent questions.

A. **Choose a specific value for the pull-back distance, x_p.**

   In order to determine the maximum possible pull-back distance, pull the car backwards until a series of noticeably louder clicks are heard. These clicks indicate that the pull-back motor is fully wound. Once the maximum pull-back distance is known, it is recommended that the chosen pull-back distance be at least half this amount. Once x_p is chosen, it should not be changed in the subsequent portions of Section 2.

B. **Place two markers on the test surface to denote $x = 0$ and $x = x_p$.**

   The idea is that the car can be placed with its nose at the x_p mark and pulled backwards until its nose is at the zero mark, which will ensure that the pull-back distance x_p is implemented correctly.

C. **Wind up the pull-back car using the pull-back distance chosen in Step A.**

   Be careful to place the nose of the car at the x_p marker and pull the car back until its nose is at the zero marker. Deviations from this procedure will lead to inaccuracies.

D. **Keeping one hand in contact with the front of the car, allow the car to make a series of short advances until the motor has completely wound down.**

   This process will take a little while and the key is to regularly stop the car so that it is unable to coast after the torque from the pull-back motor is exhausted. It may turn out that the test surface is too short for the chosen pull-back distance. If necessary, return to step A and select a smaller pull-back distance. In any case, it may take several attempts to perform this step smoothly.

E. **Place a mark at the location of the pull-back car once it has fully wound down.**

   The marker should be placed at the nose of the car to be consistent with the earlier markings.

F. **Measure the distance between the two marks, which provides an estimate of x_w.**

Repeat the procedures from Step C two or three times, making sure the results are consistent. It would be wise to use a different mark for the final position of each trial, e.g., a piece of tape with the trial number written on it.

Questions:

1. How consistent were the measurements of x_w in the different trials? Did anything happen during your trials that could have affected the estimate, e.g., did the wheels slip at any point?

2. Using the most reliable measurements of x_w from your trials, what is the ratio of x_w to x_p?

2.2 Data Collection for Estimation of M_p, k_1, and k_2

In order to estimate the constants M_p, k_1, and k_2, it is necessary to gather data for the pull-back car while it is in motion. This will be accomplished by extracting position data from the frames of a video recording in which the wound-up car accelerates from rest and then begins to decelerate. The position data can then be compared with the predictions of the mathematical model for any choice of the constants M_p, k_1, and k_2, which can be tuned to minimize the squared error of the model.

Many pull-back cars can travel upwards of 5 meters when wound up fully, which makes it almost impossible to produce a single video that captures the full distance traveled. In order to get around this difficulty, students are encouraged to use separate recordings to capture the acceleration and deceleration phases of the car. It is probably also a good idea to make at least two recordings for each phase.

Students who do not have access to a pull-back toy have two options at this stage.

(1) Use the data provided in Table 1 which was generated using the car of Figure 1. Three data sets are provided and each utilizes a frame rate of 30 frames per second. The first two experiments represent acceleration from rest, while the third captures the tail end of a run after the car has surpassed the wind-down distance. The pull-back distance for the acceleration experiments is 25 centimeters and the corresponding wind-down distance is 150 centimeters. Students who choose this option should skip the Student Task below and proceed to the questions that follow.

(2) Use the three videos that are included with the Student materials, each of which uses a frame rate of 30 frames per second. These experiments employ a pull-back truck with a pull-back distance of 20 centimeters and a wind-down distance of 218 centimeters. Students who choose this option should proceed to Step C of the Student Task below and then answer the questions that follow.

Student Task: Use the following procedure to collect data for the estimation of M_p, k_1, and k_2.

A. **Place a long strip of masking tape on the test surface and mark the tape at regular intervals, e.g., every 5 centimeters.**

There are a variety of ways to make sure that recordings of the car's motion will contain usable position data. In some cases it may be possible to run the toy car alongside a measuring stick. It might also be convenient to place small pieces of tape on the test surface every 5 centimeters instead of marking a long piece of tape.

B. **Use the video camera to record multiple experiments with the car.**

It is important that at least one experiment captures the acceleration of the car from rest in order to accurately estimate M_p and k_1. Similarly, it is necessary to have at least one recording in which the car decelerates substantially, in order to obtain a good estimate of k_2. It is a good idea to keep the camera as still as possible during the recording, which will make it easier to determine the position of the car in each frame of the video. Ideally, students will collect 3–4 videos in this step.

C. **Extract position data from your videos using a computer.**

The goal is to record the position of the front of the pull-back car and the corresponding video frame number for a sufficient number of frames. In acceleration videos, frame 0 should correspond to the moment the car is released from the starting mark. In deceleration videos, frame 0 can correspond to any frame after the car has passed the wind-down distance x_w. The time corresponding to each frame is computed as follows. Many digital cameras record 30 frames per second, in which case the time for each data point can be calculated as: `time` = `(frame number)`/30. Make sure to verify the frame rate and adjust this formula accordingly, as some cameras use 60 or 120 frames per second.

Table 1. Time and Position Data for the Pull-Back Car Depicted in Figure 1.

Frame	Distance (cm)	Frame	Distance (cm)	Frame	Distance (cm)
0	0	0	0	0	275
3	2	3	2	3	289
6	7	6	9	6	302
9	19	9	20	9	313
12	32	12	35	12	326
15	48	15	53	15	336
18	68	18	75	18	347
21	90	21	98	21	357
24	113	24	124	24	367
27	135	27	151	27	377
30	157	30	175	30	385
33	180	33	200	33	394
				36	402

Questions:

1. Estimate the velocity of the car at each time in your collected data as follows. Use a one-step formula of the form
$$v_n = \frac{x_n - x_{n-1}}{t_n - t_{n-1}}$$
for the first and last data points, but use a two-step formula of the form
$$v_n = \frac{x_{n+1} - x_{n-1}}{t_{n+1} - t_{n-1}}$$
for the remaining data points.

2. Looking at the velocity data for one of your acceleration data sets, do the observed velocities increase steadily throughout the data set? Watch out for flat spots in the velocity data, which could result from slippage of the driven wheels on the car.

3. Looking at the velocity data for one of your deceleration data sets, do the observed velocities decrease steadily throughout the data set? Are there any unusual fluctuations?

2.3 Estimation of M_p, k_1, and k_2

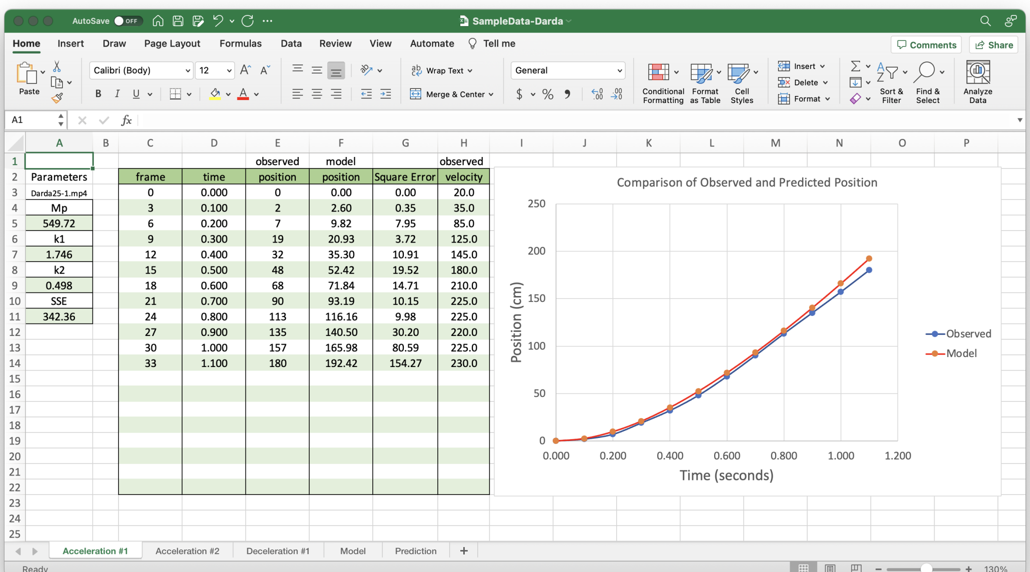
At this point, students should have 3–4 sets of time and position data for a pull-back car with at least one set corresponding to acceleration from rest and at least one set corresponding to deceleration.

The next task involves fitting the mathematical model of Section 1 to these data sets to obtain estimates for M_p, k_1, and k_2. It is critical that any acceleration data begins with the car at $x = 0$ and, for both acceleration and deceleration data sets, the corresponding model must be correctly employed. These expressions were derived in the question portion at the end of Section 1, where Question 1 identifies the form of $x(t)$ for $0 \leq x \leq x_w$ and Question 2 led to an expression for $x(t)$ when $x \geq x_w$.

Student Task: Use a spreadsheet to estimate M_p, k_1, and k_2 from the experimental data collected earlier.

A. **For each data set, create columns for all of the required variables and enter the observed time and position data.**

   Five columns are needed: (A) Frame, (B) Time t, (C) Position $x(t)$-observed, (D) Position $x(t)$-predicted, (E) Squared Error. (See Figure 3 for a screenshot.)



Parameters
Darda25-1.mp4
Mp
549.72
k1
1.746
k2
0.498
SSE
342.36

frame	time	observed position	model position	Square Error	observed velocity
0	0.000	0	0.00	0.00	20.0
3	0.100	2	2.60	0.35	35.0
6	0.200	7	9.82	7.95	85.0
9	0.300	19	20.93	3.72	125.0
12	0.400	32	35.30	10.91	145.0
15	0.500	48	52.42	19.52	180.0
18	0.600	68	71.84	14.71	210.0
21	0.700	90	93.19	10.15	225.0
24	0.800	113	116.16	9.98	225.0
27	0.900	135	140.50	30.20	220.0
30	1.000	157	165.98	80.59	225.0
33	1.100	180	192.42	154.27	230.0

Figure 3. Screenshot of the initial spreadsheet setup.

B. **Create individual cells for: M_p, k_1, k_2, and the sum of the squared error (SSE) over all data sets.**

   Start with initial parameter values $M_p = 500$, $k_1 = 1.5$, and $k_2 = 0.75$. It would be a good idea to pull the values of these constants from a single location in the spreadsheet, so that changes automatically propagate to each data set.

C. **For each data set, implement a formula for the predicted position using your work from Section 1.**

The formula for the position in the acceleration phase was derived in Question 1 of Section 1, while the formula for the position in the deceleration phase was derived in Question 2 of Section 1. The initial position and velocity for the acceleration phase should both equal zero, while the initial position and velocity for the deceleration phase should be determined from the data.

D. **Add a scatter plot of the observed and predicted values of $x(t)$.**

E. **Adjust the parameters M_p, k_1, and k_2 to improve the accuracy of the model.**

Make small adjustments to the parameters while watching the graphs and the sum of the squared error. Think about how each change will affect the graph before implementing it. If the spreadsheet is equipped with a solver, feel free to use the solver to optimize the values of M_p, k_1, and k_2 so that the sum of the squared errors across all of the data sets is minimized. Keep in mind the fact that M_p and k_1 only affect the acceleration portion of the model, while k_2 affects only the deceleration portion.

Questions:

1. What are the optimal values for M_p, k_1, and k_2?

2. How well does the mathematical model fit the data? Is there a big difference between k_1 and k_2?

3. Use the mathematical model of Section 1 to compute the maximum velocity and total distance traveled for the pull-back car using the pull-back distance that was adopted for the data collection.

3 Model Predictions

The goal of this section is to use the mathematical model to predict the maximum velocity and distance traveled as a function of the pull-back distance. The model validation in Section 2.3 used a specific pull-back distance that was chosen based on characteristics of the pull-back car used as well as the space available for testing. If another pull-back distance is used, how far is the car expected to travel? The mathematical model of Section 1 can be used to provide an approximate answer to this question.

Hooke's law for a spring of stiffness k states that the force of resistance is proportional to the deformation of the spring, i.e., $F(x) = -kx$. Consequently, the work done to compress a spring from $x = 0$ to $x = \delta$ is given by

$$\text{Work} = \int_0^\delta kx\,dx = \frac{1}{2}k\delta^2.$$

The same principle can be employed with the constant-torque model for the motion of a pull-back car. The idea is that the total energy stored in the motor should be proportional to the square of the pull-back distance or, equivalently, to the square of the wind-down distance. Recall that x_p and x_w, respectively, represent the pull-back and wind-down distances used throughout Section 2. The constant-torque model leads to the following representation of the energy stored in the spring,

$$E(x_p) := \int_0^{x_w} M(x)\, dx = M_p x_w,$$

where M_p is the quantity estimated in Section 2 that varies with x_p. Similarly, for another pull-back distance, $\tilde{x}_p$, the energy stored in the spring would be

$$E(\tilde{x}_p) := \int_0^{\tilde{x}_w} \tilde{M}(x)\, dx = \tilde{M}_p \tilde{x}_w,$$

where $\tilde{x}_w$, $\tilde{M}(x)$, and $\tilde{M}_p$ are the related quantities associated with $\tilde{x}_p$. Meanwhile, the physical intuition provided by Hooke's law suggests that the energy stored in the internal spring should be proportional to the square of the wind-down distance, from which it follows that

$$\frac{E(\tilde{x}_p)}{E(x_p)} = \frac{\tilde{x}_w^2}{x_w^2}. \tag{3}$$

Using the earlier expressions for $E(\tilde{x}_p)$ and $E(x_p)$, it is now possible to solve for $\tilde{M}_p$ in terms of M_p, x_w, and $\tilde{x}_w$. In particular, (3) is equivalent to

$$\frac{\tilde{M}_p \tilde{x}_w}{M_p x_w} = \frac{\tilde{x}_w^2}{x_w^2},$$

leading to

$$\tilde{M}_p = M_p \left(\frac{\tilde{x}_w}{x_w}\right) \qquad \text{or} \qquad \tilde{M}_p = M_p \left(\frac{\tilde{x}_p}{x_p}\right), \tag{4}$$

since the ratio of the two pull-back distances must equal the ratio of the two wind-down distances. The identity (4) makes it possible to extend the model found in Section 2 to predict the velocity and distance traveled for any pull-back distance.

Student Task: Explore the effect of pull-back distance on the distance traveled.

A. **Choose a pull-back distance $\tilde{x}_p$ that differs from the one used in Section 2.**

Pull-back cars do not necessarily work well when the pull-back distance is very small, so it may be best to add or subtract 5-10 centimeters from the original pull-back distance.

B. **Determine the corresponding mathematical model.**

It is assumed that the friction constants k_1 and k_2 do not vary with the pull-back distance, that the corresponding wind-down distance $\tilde{x}_w$ can be found using the ratio in Section 2.1, and that the corresponding torque $\tilde{M}_p$ can be computed from (4). Thus, the adjusted mathematical model can be found without any additional experimental data.

C. **Experimentally measure the distance traveled using the chosen pull-back distance.**

Conduct several experiments using the chosen pull-back distance $\tilde{x}_p$, measuring the total distance traveled by the car in each experiment. Record these values for comparison with the prediction made by the mathematical model.

Questions:

1. Use the mathematical model to predict the maximum velocity and distance traveled with pull-back distance $\tilde{x}_p$.

2. How far did the car actually travel with pull-back distance $\tilde{x}_p$? How does this compare with the findings of Question 1?

REFERENCES

[1] Wikipedia article. 2021. 'Pullback motor'. `https://en.wikipedia.org/wiki/Pullback_motor`. Accessed 10 June 2021.

[2] Brody Dylan Johnson. 2022. 3-099-S-PullBack-ModelingScenario. SIMIODE, QUBES Educational Resources. `https://qubeshub.org/community/groups/simiode/publications?id=3007`.