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STUDENT VERSION
AN INTERACTIVE ILLUSTRATION OF SINGULAR PERTURBATION FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract:

This paper introduces the basics of singular perturbation methods for solving ordinary differential equations. Interactive examples will show how the smallness of physical parameters can drastically change the nature of the solutions if those parameters were set to zero, as is the case with regular perturbation. Interestingly, the examples will illustrate the appearance of *boundary layers* as the parameters approach small values. Singular perturbation adds to the arsenal of tools that an analyst can employ to tackle problems that are influenced by a small parameter, such as those encountered in viscous flows and control engineering.

SCENARIO DESCRIPTION

The paper presents a number of ordinary differential equations, each containing a small parameter denoted by ε. These equations are presented in a series of activities that highlight the inadequacy of regular perturbation and the need to develop solutions based on singular perturbation. More precisely, the equations can be described as

$$F(x, y, y', y'', \dots, y^{(n-1)}, y^{(n)}, \varepsilon) = 0, \qquad x \in \mathcal{I} \tag{1}$$

where x is the independent variable, defined in the interval $\mathcal{I}$, y is the dependent variable, and ε is the small parameter ($\varepsilon \ll 1$).

In regular perturbation, a power series of the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots \tag{2}$$

is sought to approximate the solution to (1). As will be seen in the following activities, this naive approach will not work for the type of problems discussed in this paper.

To help the reader to appreciate the power of singular perturbation as an analysis tool, an accompanying electronic spreadsheet has been developed to facilitate the exploration of solutions by dynamically controlling the value of the small parameter ε. The treatment in this paper is very elementary. The interested reader can refer to [1]–[3] for a more in-depth treatment of perturbation theory. For an introductory presentation of regular perturbation, the reader can refer to [4]; and for an introductory treatment of singular perturbation, the reader can refer to [5].

Activity 1: Boundary Value Problem (BVP)

Consider the BVP

$$4\varepsilon y'' + 4y' - y = 0, \qquad y(0) = 0,\ y(1) = 1 \tag{3}$$

Task 1.1: Obtaining the Exact Solution

Verify that the solution to (3) is given by

$$y(x) = \frac{1}{e^{r_1} - e^{r_2}} e^{r_1 x} - \frac{1}{e^{r_1} - e^{r_2}} e^{r_2 x} \tag{4}$$

where

$$r_1 = \frac{-1 + \sqrt{1+\varepsilon}}{2\varepsilon} \quad \text{and} \quad r_2 = \frac{-1 - \sqrt{1+\varepsilon}}{2\varepsilon} \tag{5}$$

The exact solution will be used as the benchmark for comparing the goodness of the approximate solution obtained by singular perturbation. Bear in mind that most equations do not have closed-form solutions; in those cases, numerical methods may be required if comparisons are needed.

Task 1.2: Implications of Setting the Small Parameter to Zero

Analyze the problem in (3) when ε is set to zero. Notice that ε multiplies the highest-order derivative. Carefully consider the implications on the character of the solution and the boundary conditions, if $\varepsilon = 0$. This will demonstrate that regular perturbation as in (2) is inadequate because the *unperturbed* solution $y_0(x)$ for (3) satisfying both boundary conditions does not exist when $\varepsilon = 0$. This makes (3) a *singular perturbation problem.*

Task 1.3: Boundary Layer

An interesting phenomenon occurs as the parameter ε becomes increasingly small. To see this, the reader can use the macro-enabled spreadsheet SingularPerturbation-EXC-StudentVersion.xlsm to observe the behavior of the solution to (3) as ε approaches zero. Figure 1 shows a screen capture of the first sheet of the Excel file. The user can slide the control bar to vary the parameter ε, whose current value is displayed in cell E4. In this case, the sheet shows tabulated values of the exact solution and the corresponding graph for $\varepsilon = 0.02$.

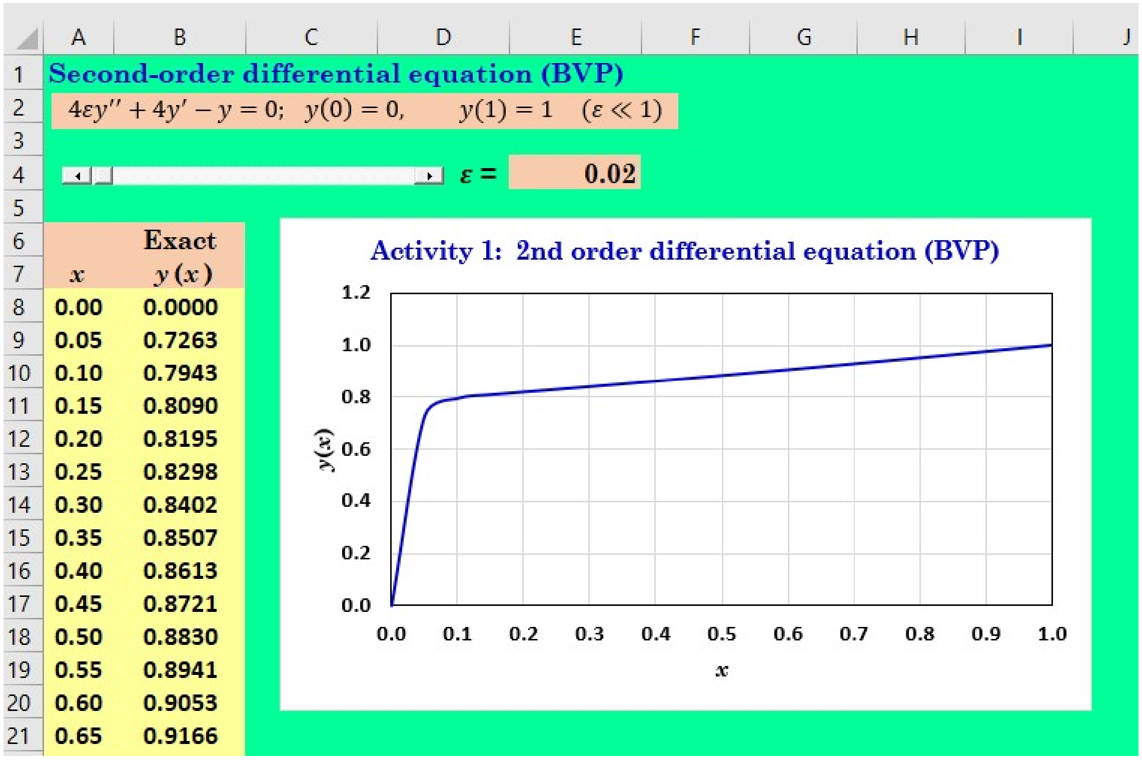


Figure 1. Microsoft Excel spreadsheet user interface for exploring solutions to (3) as the user varies the parameter ε by sliding the control bar. The current value of ε is displayed in cell E4.

It can be observed that as the parameter ε approaches zero, the solution exhibits an abrupt change. This small interval near $x = 0$ is known as the *boundary layer*. On the other hand, the solution near the boundary at $x = 1$ seems to vary more slowly; and will be referred to as the *outer layer*. This suggests splitting the solution into two: an *outer solution* that is valid for $x = O(1)$ and an *inner solution* that is valid for $x = O(\varepsilon)$, with the appropriate length scaling of the boundary layer.

After completing this activity, the reader should have a good understanding of the behavior of the solution to (3) as the parameter ε becomes increasingly small. The next activity introduces the main ideas for constructing approximate solutions based on singular perturbation.

Activity 2: Singular Perturbation Method

Again, consider the BVP in (3)

$$4\varepsilon y'' + 4y' - y = 0, \qquad y(0) = 0,\ y(1) = 1\,.$$

Attempting to find an approximation of the form given in (2) will not work, as the reader may have concluded from Activity 1. Nevertheless, the following tasks outline the steps for obtaining an approximate solution using singular perturbation.

Task 2.1: Obtaining the Outer Solution

Substitute (2) into (3) and then obtain a set of differential equations by collecting terms in powers of ε, i.e., ε^0, ε^1, ε^2, ... Show that the following set is obtained:

$$\varepsilon^0: \qquad 4y_0' - y_0 = 0, \qquad y_0(1) = 1 \tag{6}$$
$$\varepsilon^1: \qquad 4y_0'' + 4y_1' - y_1 = 0, \qquad y_1(1) = 0 \tag{7}$$
$$\varepsilon^2: \qquad 4y_1'' + 4y_2' - y_2 = 0, \qquad y_2(1) = 0 \tag{8}$$
$$\vdots$$

In the preceding equations, the dependence of y on x has been dropped to simplify the writing. Notice that only the boundary condition at $x = 1$ is being matched. Why not match at $x = 0$?

Now solve (6)–(8) in succession and verify that

$$\begin{aligned} y_0(x) &= e^{(x-1)/4}\,, \\ y_1(x) &= \frac{1}{16}e^{(x-1)/4}(1-x)\,, \\ y_2(x) &= \frac{1}{512}e^{(x-1)/4}(x^2+14x-15)\,. \end{aligned}$$

Therefore, the second-order approximation (matched at $x = 1$) is given by

$$y_{\text{outer}}(x) = e^{(x-1)/4} + \varepsilon\left(\frac{1}{16}e^{(x-1)/4}(1-x)\right) + \varepsilon^2\left(\frac{1}{512}e^{(x-1)/4}(x^2+14x-15)\right) + O(\varepsilon^3)\,. \tag{9}$$

The preceding approximation is referred to as the *outer solution* or *outer layer.* The outer solution may be thought of as the layer outside the boundary layer. As can be seen, obtaining this approximation follows the same steps as in regular perturbation, except that the boundary condition is imposed at one end only.

Task 2.2: Obtaining the Inner Solution

To obtain the solution inside the boundary layer, notice that it is important to keep the highest-order derivative term in

$$4\varepsilon y'' + 4y' - y = 0, \qquad y(0) = 0,\; y(1) = 1\,,$$

to be able to impose the condition at the boundary layer at $x = 0$. To find this solution, re-scale the variable x as indicated by

$$x = \varepsilon^\alpha X, \qquad Y(X) = y(\varepsilon^\alpha X), \qquad \frac{d}{dx} = \varepsilon^{-\alpha}\frac{d}{dX}, \qquad \frac{d^2}{dx^2} = \varepsilon^{-2\alpha}\frac{d^2}{dX^2}\,, \tag{10}$$

where α is a parameter to be determined later.

Using the length scaling in (10), the original differential equation (3), with boundary condition matched at $x = 0$, is transformed into

$$4\varepsilon^{1-2\alpha}Y'' + 4\varepsilon^{-\alpha}Y' - Y = 0, \qquad Y(0) = 0\,, \tag{11}$$

where $(\cdot)'$ in (11) denotes differentiation with respect to X.

To determine α, perform a *dominant balance analysis.* This analysis will reveal the dominant pair in (11) and, hence, the terms that can be neglected because they would be comparatively much smaller. In this case, there are three possible pairs to investigate:

- Suppose $4\varepsilon^{1-2\alpha}Y'' \sim Y$ is the dominant balance as $\varepsilon \to 0$. Since $4\varepsilon^{1-2\alpha}Y'' = O(\varepsilon^{1-2\alpha})$ and $Y = O(1)$ as $\varepsilon \to 0$, this dominant balance would be possible if $1 - 2\alpha = 0$ or $\alpha = 1/2$. However, this would render $4\varepsilon^{-\alpha}Y' = O(\varepsilon^{-1/2}) \gg 4\varepsilon^{1-2\alpha}Y'' = O(1)$, which contradicts the original assumption of the dominant balance.
- Suppose $4\varepsilon^{-\alpha}Y' \sim Y$ is the dominant balance as $\varepsilon \to 0$. Since $4\varepsilon^{-\alpha}Y' = O(\varepsilon^{-\alpha})$ and $Y = O(1)$ as $\varepsilon \to 0$, this dominant balance would be possible if $\alpha = 0$. But this would make $4\varepsilon^{1-2\alpha}Y'' = O(\varepsilon)$ the smallest term and neglected. This is not desirable because the highest-order derivative is to be retained. Actually, this case was trivial and the conclusion should not be surprising. This is precisely what was done to find the outer approximation!
- Suppose $4\varepsilon^{1-2\alpha}Y'' \sim 4\varepsilon^{-\alpha}Y'$ is the dominant balance as $\varepsilon \to 0$. Since $4\varepsilon^{1-2\alpha}Y'' = O(\varepsilon^{1-2\alpha})$ and $4\varepsilon^{-\alpha}Y' = O(\varepsilon^{-\alpha})$ as $\varepsilon \to 0$, this dominant balance would be possible if $1 - 2\alpha = -\alpha$ or $\alpha = 1$. This implies that $Y = O(1) \ll 4\varepsilon^{1-2\alpha}Y'' = O(\varepsilon^{-1})$ as $\varepsilon \to 0$, which is consistent with the initial assumption. This is the correct dominant balance.

The previous analysis leads to $\alpha = 1$ (with $4\varepsilon^{1-2\alpha}Y'' \sim 4\varepsilon^{-\alpha}Y'$) and the following length scaling

$$x = \varepsilon X \qquad \text{and} \qquad Y(X) = y(\varepsilon X)\,. \tag{12}$$

With the scaling given in (12), show that the original equation becomes

$$4Y'' + 4Y' - \varepsilon Y = 0, \qquad Y(0) = 0\,, \tag{13}$$

where $(\cdot)'$ is now differentiation with respect to X.

Observe that (13) is now amenable to regular perturbation. In particular, the leading-order approximation ($\varepsilon = 0$) results in

$$Y'' + Y' = 0, \qquad Y(0) = 0\,, \tag{14}$$

which has the general solution

$$Y_{\text{inner}}(X) = C_0\left(1 - e^{-X}\right)\,, \tag{15}$$

or, since $x = \varepsilon X$,

$$y_{\text{inner}}(x) = C_0\left(1 - e^{-x/\varepsilon}\right)\,. \tag{16}$$

The arbitrary constant C_0 will be determined in the next task.

Task 2.3: Asymptotic Matching

Recall that an approximate solution is sought for the BVP

$$4\varepsilon y'' + 4y' - y = 0, \qquad y(0) = 0,\ y(1) = 1\,.$$

Referring to (9) and (16), the main results from the previous two tasks can be summarized as

$$\begin{aligned} y_{\text{outer}}(x) &= e^{(x-1)/4}, && x = O(1)\,, \\ y_{\text{inner}}(x) &= C_0\left(1 - e^{-x/\varepsilon}\right), && x = O(\varepsilon)\,, \end{aligned} \tag{17}$$

To find the arbitrary constant C_0, apply the *matching condition* to find the common limit y^*

$$y^* = \lim_{x \to 0^+} y_{\text{outer}}(x) = \lim_{x \to \infty} y_{\text{inner}}(x)\,, \tag{18}$$

to obtain $y^* = e^{-1/4} = C_0$.

The matching condition (18) states that the *outer solution*, as the *outer variable* moves toward the *boundary layer*, must equal the *inner solution*, as the *inner variable* moves toward the *outer layer*. In applying the matching condition, one needs to be careful about the location of the boundary layer. In (18), it is assumed that the boundary layer is adjacent to 0; if this is not the case, adjust the limits accordingly.

Thus, the outer and inner solutions are given by, respectively,

$$\begin{aligned} y_{\text{outer}}(x) &= e^{(x-1)/4}, && x = O(1)\,, \\ y_{\text{inner}}(x) &= e^{-1/4}\left(1 - e^{-x/\varepsilon}\right), && x = O(\varepsilon)\,. \end{aligned} \tag{19}$$

Finally, obtain the composite solution by constructing a *uniform approximation*, $y_{\text{unif}}(x)$, for the solution by adding the outer and inner approximations and subtracting the common limit, i.e.,

$$y_{\text{unif}}(x) = y_{\text{outer}}(x) + y_{\text{inner}}(x) - y^*\,. \tag{20}$$

In this example,

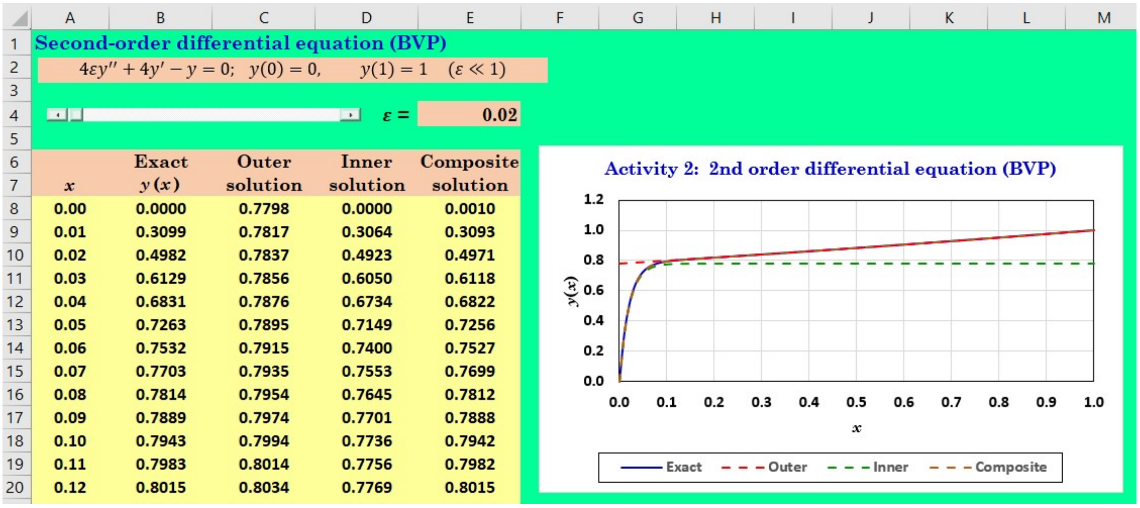
$$y_{\text{unif}}(x) = e^{-1/4}\left(e^{x/4} - e^{-x/\varepsilon}\right), \qquad 0 \le x \le 1\,. \tag{21}$$

Figure 2 shows a screen capture of the accompanying spreadsheet displaying the outer, inner, and composite approximations along with the exact solution. The graphs are shown for $\varepsilon = 0.02$. For this small value of the parameter ε, it can be observed that both the inner and outer approximations agree with the exact solution fairly well in their respective layers; the composite solution is almost indistinguishable from the exact solution for this value of ε. The user can slide the control bar to change the value of this parameter; for greater values of ε a slight discrepancy between the composite and exact solutions can be noticed.

Activity 3: Singular Perturbation for Nonlinear Differential Equation

Obtain the leading-order asymptotic approximation to the following nonlinear BVP

$$\varepsilon y'' + (1+\varepsilon)y' + e^{-y} = 0, \qquad y(0) = 2,\ y(1) = 0\,. \tag{22}$$



x	Exact $y(x)$	Outer solution	Inner solution	Composite solution
0.00	0.0000	0.7798	0.0000	0.0010
0.01	0.3099	0.7817	0.3064	0.3093
0.02	0.4982	0.7837	0.4923	0.4971
0.03	0.6129	0.7856	0.6050	0.6118
0.04	0.6831	0.7876	0.6734	0.6822
0.05	0.7263	0.7895	0.7149	0.7256
0.06	0.7532	0.7915	0.7400	0.7527
0.07	0.7703	0.7935	0.7553	0.7699
0.08	0.7814	0.7954	0.7645	0.7812
0.09	0.7889	0.7974	0.7701	0.7888
0.10	0.7943	0.7994	0.7736	0.7942
0.11	0.7983	0.8014	0.7756	0.7982
0.12	0.8015	0.8034	0.7769	0.8015

Figure 2. Spreadsheet showing inner, outer, and composite solutions to the BVP in (3). The current value of ε is displayed in cell E4.

This nonlinear equation does not have a closed-form solution. If it is desired to compare the approximation obtained by singular perturbation, one may resort to a numerical approach like the shooting method for boundary value problems or Runge-Kutta for initial value problems. As the reader will be about to see, the resulting leading-order equations do have analytical solutions. Therein lies the usefulness of perturbation methods, as these provide tools for analysis to gain valuable insights, which sometimes are obscured by numerical approaches.

Task 3.1: Obtain the Outer Solution

To obtain the outer solution to (22), set $\varepsilon = 0$ and solve

$$y' + e^{-y} = 0, \qquad y(0) = 2,\ y(1) = 0\,. \tag{23}$$

The general solution to (23) is

$$y_0(x) = \ln(C - x)\,,$$

where C is a constant. To determine C, assume the outer layer is adjacent to $x = 1$. If this is the case, then $C = 2$ and, consequently, the leading-order approximation for the outer solution will be

$$y_{\text{outer}}(x) = \ln(2 - x), \qquad x = O(1)\,. \tag{24}$$

In finding the outer solution, the reader should notice that the outer layer cannot be adjacent to $x = 0$. Why?

Task 3.2: Obtain the Inner Solution

Before finding the dominant balance, rewrite (22) as

$$\varepsilon y'' + y' + \varepsilon y' + e^{-y} = 0, \qquad y(0) = 2,\ y(1) = 0\,. \tag{25}$$

Since there are four terms in (25), there are six possible pairs to investigate for dominant balance. Proceeding similarly as in Task 2.2 with the scaling (10), the reader should conclude that $\varepsilon y'' \sim y'$ is the dominant balance with $\alpha = 1$, $x = \varepsilon X$, and $Y(X) = y(\varepsilon X)$. This means that (25) can be rewritten as

$$Y'' + Y' + \varepsilon Y' + \varepsilon e^{-Y} = 0, \qquad Y(0) = 2,\ Y(1/\varepsilon) = 0\,. \tag{26}$$

Setting $\varepsilon = 0$ in (26), the inner solution to leading-order is given by

$$Y(X) = C_1 e^{-X} + C_2\,.$$

Imposing $Y(0) = 2$, where the boundary layer is adjacent to, gives $C_2 = -C_1 + 2$ and, therefore,

$$Y(X) = C_1 e^{-X} - C_1 + 2\,,$$

or

$$y_{\text{inner}}(x) = C_1 e^{-x/\varepsilon} - C_1 + 2\,, \tag{27}$$

where C_1 is a constant to be determined by asymptotic matching.

Task 3.3: Obtain the Uniform Approximation

Using the matching condition (18) yields the common limit

$$y^* = \lim_{x\to 0} \ln(2-x) = \lim_{x\to\infty} \Big(C_1 e^{-x/\varepsilon} - C_1 + 2\Big) \qquad \Longrightarrow \qquad y^* = \ln 2, \quad C_1 = 2 - \ln 2\,. \tag{28}$$

Hence, the outer and inner solutions to leading order are, respectively,

$$\begin{aligned} y_{\text{outer}}(x) &= \ln(2-x), && x = O(1)\,, \\ y_{\text{inner}}(x) &= (2-\ln 2)e^{-x/\varepsilon} + \ln 2, && x = O(\varepsilon)\,. \end{aligned} \tag{29}$$

Finally, the uniform approximation of the solution to (22) is obtained by adding the outer and inner solutions and subtracting the common limit as indicated in (20)

$$y_{\text{unif}}(x) = \ln(2-x) + (2-\ln 2)e^{-x/\varepsilon}, \qquad 0 \le x \le 1\,. \tag{30}$$

Figure 3 shows the graphs for the outer, inner, and composite solutions to the boundary value problem in (22) when $\varepsilon = 0.05$. The value of the parameter may be adjusted by the user by simply sliding the control bar. The graphs confirm that the boundary layer is located near $x = 0$. The composite solution agrees well with the inner solution in a narrow interval around $x = 0$; likewise, the composite solution agrees well with the outer solution in the region outside the boundary layer.

Nonlinear differential equation (BVP)

$\varepsilon y'' + (1+\varepsilon)y' + e^{-y} = 0, \quad y(0) = 2, \quad y(1) = 0 \quad (\varepsilon \ll 1)$

$\varepsilon =$ **0.05**

x	Outer solution	Inner solution	Composite solution
0.00	0.6931	2.0000	2.0000
0.01	0.6881	1.7631	1.7581
0.02	0.6831	1.5692	1.5591
0.03	0.6780	1.4104	1.3952
0.04	0.6729	1.2804	1.2602
0.05	0.6678	1.1739	1.1486
0.06	0.6627	1.0868	1.0563
0.07	0.6575	1.0154	0.9798
0.08	0.6523	0.9570	0.9162
0.09	0.6471	0.9092	0.8631
0.10	0.6419	0.8700	0.8187
0.11	0.6366	0.8380	0.7814
0.12	0.6313	0.8117	0.7498

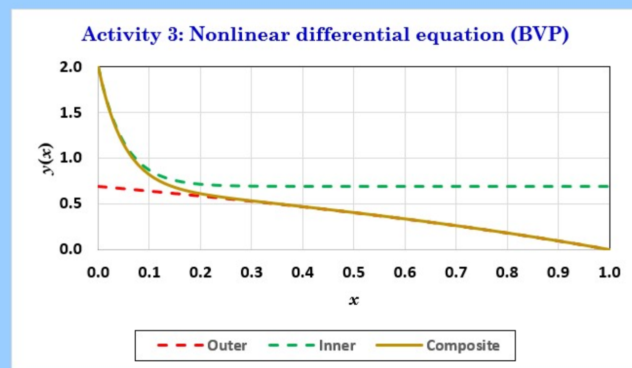


Figure 3. Spreadsheet showing inner, outer, and composite solutions to the nonlinear BVP in (22). The current value of ε is displayed in cell E4.

CONCLUDING REMARKS

This paper presented an introduction to the method of singular perturbation. The activities included examples that can be categorized as boundary layer problems. Due to the singular nature of these problems, a scaling of the independent variable was introduced to split the solution into two parts: a rapidly varying solution that is valid within the boundary layer; and a slowly varying solution that is valid in the outer layer.

An accompanying macro-enabled Excel workbook was developed to allow the user to explore the behavior of the solutions as the value of the small parameter changes. These solutions are presented both in tabular form and in graphical form.

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