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SIMIODE A SYSTEMIC INITIATIVE FOR MODELING INVESTIGATIONS & OPPORTUNITIES WITH DIFFERENTIAL EQUATIONS

# Modeling Falling Column of Water

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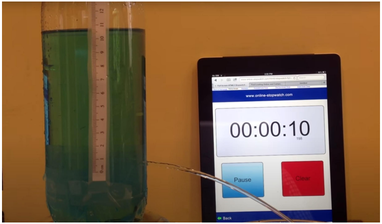
# Modeling Falling Column of Water

We collect data on a falling column of water and model the height using first principles from physics with a differential equation.

1. Video, data, qualitative behavior, empirical model
2. First principles analytic model, Torricelli's Law
3. Differential equation, estimate parameters, validate model
4. Discussions

Source:
Modeling Scenario in SIMIODE - **1-015-Torricelli**

We use data taken from video at SIMIODE YouTube Channel




**https://www.youtube.com/watch?v=NBr4DOj4OTE** .

Cylindrical column (radius = 4.17 cm) of water empties through a hole (diameter = 11/16" = 0.218281 cm) in bottom of column. Exit hole at bottom of column - height is 0 cm.

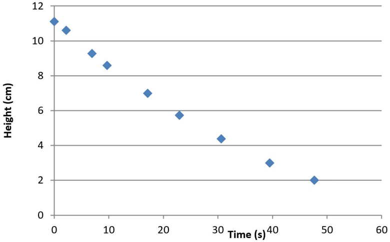
We seek to model $h(t)$, the height of the column of water.

Just visit. No run yet.

Here is data we collected. What do you see or notice?

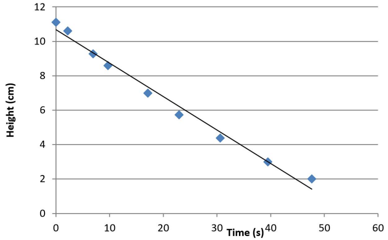
Make some observations now.

| Time (s) | Height (cm) |
|---|---|
| 0.0 | 11.1 |
| 2.187 | 10.6 |
| 6.933 | 9.3 |
| 9.717 | 8.6 |
| 17.102 | 7.0 |
| 22.968 | 5.75 |
| 30.603 | 4.4 |
| 39.503 | 3.0 |
| 47.663 | 2.0 |



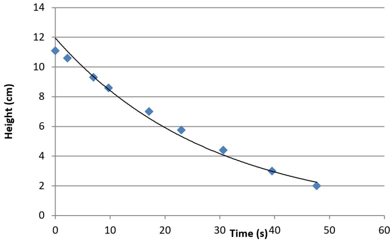
# Linear Fit?

| Time (s) | Height (cm) |
|---|---|
| 0.0 | 11.1 |
| 2.187 | 10.6 |
| 6.933 | 9.3 |
| 9.717 | 8.6 |
| 17.102 | 7.0 |
| 22.968 | 5.75 |
| 30.603 | 4.4 |
| 39.503 | 3.0 |
| 47.663 | 2.0 |



# Exponential Decay Fit?

| Time (s) | Height (cm) |
|---|---|
| 0.0 | 11.1 |
| 2.187 | 10.6 |
| 6.933 | 9.3 |
| 9.717 | 8.6 |
| 17.102 | 7.0 |
| 22.968 | 5.75 |
| 30.603 | 4.4 |
| 39.503 | 3.0 |
| 47.663 | 2.0 |



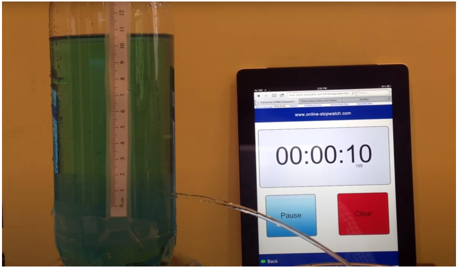
All are empirical fits with no understanding.

They just fit a function to data.

And neither line nor exponential are good.

Run video of falling column and observe.

**https://www.youtube.com/watch?v=NBr4DOj4OTE**




What happens to height $h(t)$?

How fast is column of water falling? Early and later?

From the video of the falling column what can we see?

For large $h(t)$ the column of water falls faster or slower or same ...

For small $h(t)$ falls faster or slower or same ...

| Time (s) | Height (cm) |
|---|---|
| 0.0 | 11.1 |
| 2.187 | 10.6 |
| 6.933 | 9.3 |
| 9.717 | 8.6 |
| 17.102 | 7.0 |
| 22.968 | 5.75 |
| 30.603 | 4.4 |
| 39.503 | 3.0 |
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What is your conclusion about $\dfrac{dh(t)}{dt}$?

So might something like this be true?

$$\frac{dh(t)}{dt} = f(h(t)), \quad h(0) = h_0 .$$

Where for large $h(t)$ we have large $f(h(t))$

Where for small $h(t)$ we have small $f(h(t))$

And, of course, $f(h(t))$ is negative. Why?

Let's check out change in height over various intervals of time.

| Time (s) | Height (cm) |
|---|---|
| 0.0 | 11.1 |
| 2.187 | 10.6 |
| 6.933 | 9.3 |
| 9.717 | 8.6 |
| 17.102 | 7.0 |
| 22.968 | 5.75 |
| 30.603 | 4.4 |
| 39.503 | 3.0 |
| 47.663 | 2.0 |

Check out the average rate of falling of the height of the column of water in several intervals, say, $[0, 2.187]$,

$$\frac{10.6 - 11.1}{2.187 - 0} = -0.2286\,,$$

or in the interval $[39.503, 47.663]$,

$$\frac{2.0 - 3.0}{47.663 - 39.503} = -0.122549\,.$$

What do you see? What can you say about $h'(t)$?

Let's find a model from some first principles.

This would be an analytic model.

**NOT** just fit a function to data.

**NOT** just "it looks like it falls faster or slower."

| Time (s) | Height (cm) |
|---|---|
| 0.0 | 11.1 |
| 2.187 | 10.6 |
| 6.933 | 9.3 |
| 9.717 | 8.6 |
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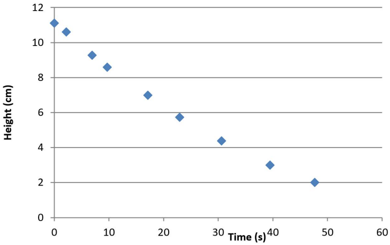


Enter Evangelista Torricelli 1608–1647, an Italian physicist and mathematician, and a student of Galileo. Best known for his invention of the barometer.

Torricelli's Law to the rescue!

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)}, \quad h(0) = h_0 \quad b > 0.$$

Say it out loud in sentence form.

Explain to yourself what it means.

Does Torricelli's Law agree with observations?

For large $h(t)$ the column of water falls faster or slower ....

For small $h(t)$ the column of water falls faster or slower ....

## Torricelli's Law

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)}, \quad h(0) = h_0 \quad b > 0.$$

Does Torricelli's Law agree with observations?

For large $h(t)$ falls **faster** or slower ... YES!

For small $h(t)$ falls faster or **slower** ... YES!

We build the model that IS Torricelli's Law from First Principles.

This will be an analytic model.

Given the cross sectional area of the cylinder of water as a function of height and the area of the tiny exit hole at the bottom of the cylinder can we model the outflow of the water from the cylinder?

We will apply this First Principle

**The Law of Conservation of Energy.**

We could ask you for a short statement of The Law of Conservation of Energy.

Thank you!

Basically, The Law of Conservation of Energy says that total energy is conserved.

We will apply it to a slab of water, first at the surface of the column of water and then at the bottom of the column ($h = 0$)

Total Energy is the the sum of the **potential energy** and the **kinetic energy** of a particle of mass $m$ and this sum is constant at each instance in time, $t$.

Care to share formula for **potential energy** and for **kinetic energy**?

Go ahead and jump the gun!

Consider mass of water $m$ initially atop a cylinder of water, some $h$ meters above exit hole, this mass has

**initial potential energy** $PE_i = m \cdot g \cdot h$, where $g$ is the acceleration due to gravity and

**initial kinetic energy** $KE_i = \frac{1}{2}mv_i^2$, where $v_i$ is the initial velocity of the mass.

Thus we have initial total energy of $TE_i$ when the mass of water is on the top of the cylinder of water:

$$TE_i = KE_i + PE_i = \frac{1}{2}mv_i^2 + mgh\,.$$

When this mass of water reaches the exit hole it has height 0 meters and a final velocity of $v_f$.

Hence, the total energy at the final time, $TE_f$, the mass reaches the exit hole where $h = 0$ is

$$TE_f = KE_f + PE_f = \frac{1}{2}mv_f^2 + mg \cdot h = \frac{1}{2}mv_f^2 + mg \cdot 0 = \frac{1}{2}mv_f^2 \, .$$

Now applying The Law Conservation of Energy, $TE_i = TE_f$, we build an equation - see the equal sign!!

Now by The Law Conservation of Energy,

$$TE_i = \frac{1}{2}mv_i^2 + mgh = \frac{1}{2}mv_f^2 = TE_f \,,$$

Divide both sides by $m$ and multiply by 2 - to solve for $v_f$:

$$v_f = \sqrt{2gh + v_i^2}\,.$$

Since $v_i = 0$ we have one classical form of Torricelli's Law

$$v_f = \sqrt{2gh}\,,$$

where $v_f$ is the speed of the water as it leaves the exit hole.

We employ an effective tool in modeling: **equate two ways of computing** loss in volume of water in tank at time $t$. **First**,

$$A(h(t))h'(t) \tag{1}$$

where $A(h(t))$ is cross sectional area of column at height $h(t)$, often constant, so $A(h(t)) = A$ in our case, and **second**

$$-v_f \cdot a \cdot \alpha = -\sqrt{2\,g\,h} \cdot a \cdot \alpha = -a \cdot \alpha \cdot \sqrt{2\,g\,h}\,. \tag{2}$$

- $v_f$ is velocity of water exiting the column of water when the water is at height $h(t)$,
- $a$ is the cross sectional area of the small bore hole through which the water exits, and
- $\alpha$ is an empirical number, percent of maximum flow rate through small hole, due to friction and constriction. $\alpha$, is called *discharge or contraction coefficient*.

Now **equating** (1) and (2) - two different ways to calculate the rate at which the volume of water in the cylinder decreases - gives rise to a working version of Torricelli's Law for the height of a constant cross section column of water, $h(t)$, at time $t$:

$$A \cdot h'(t) = A(h(t)) \cdot h'(t) = -a\alpha\sqrt{2g \cdot h(t)}\,, \tag{3}$$

where $h(t)$ is the height of the column of water at time $t$.

In our case $A(h(t)) = A$ as cross section area of water is constant.

General form of Torricelli's Law in our case

$$A \cdot h'(t) = A(h(t)) \cdot h'(t) = -a\alpha\sqrt{2g \cdot h(t)}\,,$$

Solve (3) for $h'(t)$ and note that $A(h(t)) = A$. Gather all the constants except $g$ into one big constant, $b$, we have

$$h'(t) = -b\sqrt{g \cdot h(t)}\,. \tag{4}$$

How could we use our water videos to validate this model.
Take your time, think before you share your ideas.

So we have an analytic model (differential equation!) for $h(t)$.

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)}\,, \quad h(0) = h_0\,.$$

We solve this differential equation for $h(t)$ to realize a model.

What strategy/technique can we employ? What technology?

We use this solution and our data to estimate parameter $b$ and validate our model by comparing model predictions to data.

$$\frac{dh(t)}{dt} = -b\sqrt{g \cdot h(t)} = -b\sqrt{g} \cdot (h(t))^{1/2}.$$

**Separate the variables**

$$(h(t))^{-1/2} \cdot \frac{dh(t)}{dt} = -b\sqrt{g}.$$

OR

$$(h(t))^{-1/2} \cdot dh = -b\sqrt{g} \cdot dt.$$

Integrate both sides. (What is C?)

$$\int (h(t))^{-1/2} \cdot \frac{dh(t)}{dt}\, dt = \int -b\sqrt{g}\, dt + C\,,$$

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + C.$$

Now to find $C$ using Initial Conditions:

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + C.$$

$$2(h(0))^{1/2} = -b\sqrt{g} \cdot 0 + C = C.$$

Thus we have

$$2(h(t))^{1/2} = -b\sqrt{g} \cdot t + 2(h(0))^{1/2}.$$

Divide both sides by 2 and then square both sides yields:

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2}\right)^2. \tag{5}$$

This is model for height of the column of water, $h(t)$, at time $t$.

What do we know and what do we need to estimate $b$ in (5)?

$h(0) = 11.1$ cm and $g = 980$ cm/s$^2$

Thus from $h(0) = 11.1$ cm and $g = 980$ cm/s$^2$

$$h(t) = \left(-\frac{b\sqrt{g}}{2} \cdot t + (h(0))^{1/2}\right)^2$$

becomes

$$h(t) = \left(-\frac{b\sqrt{980}}{2} \cdot t + (11.1)^{1/2}\right)^2 ,$$

and expanded in decimals we have

$$h(t) = (-15.6525 \cdot b \cdot t + 3.33166)^2 . \qquad (6)$$

We have arrived at our model and now we seek to determine $b$ and validate our model.

We turn to our Excel spreadsheet and seek to determine the parameter $b$ which minimizes the sum of the squared errors between our data $(h_i)$ and our model $(h(t_i))$ over our data points.

$$SSE(b) = \sum_{i=1}^{9} (h_i - h(t_i))^2 \ .$$

Minimize as a function of the parameter $b$:

$$SSE(b) = \sum_{i=1}^{9} (h_i - h(t_i))^2 \ .$$

where

- $t_i$ is the $i^{\text{th}}$ time observation,
- $h_i$ is the observed height at time $t_i$,
- $h(t_i)$ is our model's prediction of the height at time $t_i$, and
- $n = 9$ is the number of data points we have.

# Model Analysis in Excel Using Solver

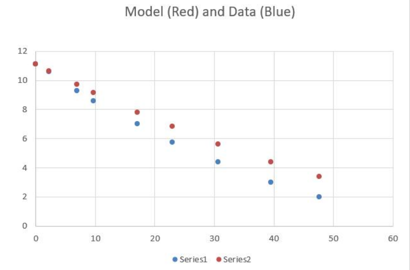
Data collected Friday, 5 August 2016 by Brian Winkel

SOURCE for Data
https://www.youtube.com/watch?v=NBr4DOj4OTE
Radius of hole 11/64" = 0.218281 cm and radius of cylinder 4.17 cm

Model h'(t) = -b Sqrt(g h(t))

Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2

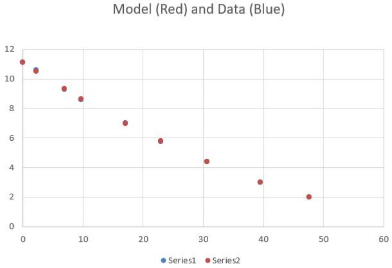
| | | | | b= | 0.002 |
|---|---|---|---|---|---|
| | **Zeroed** | **Actual** | **Model** | **SSE** | |
| **Time (s)** | **Time** | **Height (cm)** | | | |
| 8.679 | 0 | 11.1 | 11.09995836 | 1.73426E-09 | |
| 10.866 | 2.187 | 10.6 | 10.64844791 | 0.0023472 | |
| 15.612 | 6.933 | 9.3 | 9.700872913 | 0.160699092 | |
| 18.396 | 9.717 | 8.6 | 9.165570453 | 0.319869938 | |
| 25.781 | 17.102 | 7 | 7.819192408 | 0.671076202 | |
| 31.647 | 22.968 | 5.75 | 6.825923093 | 1.157610501 | |
| 39.282 | 30.603 | 4.4 | 5.634134022 | 1.523086785 | |
| 48.182 | 39.503 | 3 | 4.389102871 | 1.929606788 | |
| 56.342 | 47.663 | 2 | 3.384016994 | 1.915503039 | |
| | | | Total SEE | 7.679799546 | |




We can use Excel's Solver to minimize the TOTAL SEE or SSE which is currently 7.679799546 with parameter $b = 0.002$ by asking Solver to minimize SEE or SSE as a function of $b = 0.002$ cell.

# Parameter Estimation with Excel Solver - Results

| | | | | | |
|---|---|---|---|---|---|
| Model h'(t) = -b Sqrt(g h(t)) | | | | | |
| | | | | | |
| Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2 | | | | | |
| | | | | b= | 0.002581 |
| | | | | | |
| | **Zeroed** | **Actual** | **Model** | **SSE** | |
| **Time (s)** | **Time** | **Height (cm)** | | | |
| 8.679 | 0 | 11.1 | 11.09995836 | 1.73426E-09 | |
| 10.866 | 2.187 | 10.6 | 10.51908638 | 0.006547014 | |
| 15.612 | 6.933 | 9.3 | 9.312232219 | 0.000149627 | |
| 18.396 | 9.717 | 8.6 | 8.638501405 | 0.001482358 | |
| 25.781 | 17.102 | 7 | 6.973871293 | 0.000682709 | |
| 31.647 | 22.968 | 5.75 | 5.778477118 | 0.000810946 | |
| 39.282 | 30.603 | 4.4 | 4.390799244 | 8.46539E-05 | |
| 48.182 | 39.503 | 3 | 3.013347567 | 0.000178158 | |
| 56.342 | 47.663 | 2 | 1.977592125 | 0.000502113 | |
| | | | Total SEE | 0.010437581 | |




Note: TOTAL SEE was at 7.679799546.

**Go live to Excel.**

## Assignment

1. Write an overview of the modeling process to obtain $h'(t)$ using first principles - not all details, just highlights. Arrive at the model

$$h(t) = \left(-\frac{b\sqrt{g}}{2}\cdot t + (h(0))^{1/2}\right)^2 .$$

2. Collect data for your team's cylinder from

   **https://www.simiode.org/resources/488** .

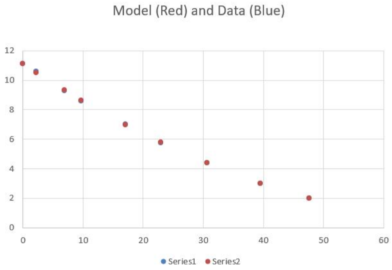
   What if we took many data points? Few data points? Try both by taking a subset of ALL the data points you took for "few data" points and see how your parameter $b$ fares.

In Excel we do these steps to determine best fit parameter $b$:

3. Compute our model value of the height $h(t_i)$ at time $t_i$.
4. Take the differences between actual data and model prediction, i.e. $h_i - h(t_i)$, and square these differences, $(h_i - h(t_i))^2$.
5. Sum these square errors to obtain $SSE(b)$.
6. Use Excel's Solver to minimize $SSE(b)$.
7. Read the value of $b$ and put it in our model as best parameter estimate of $b$.
8. Plot our best model values on the same axes as our data and compare.
9. Collect the parameters $b$ for the various outflow hole sizes from different videos selected by teams and see if there is any relationship between hole size and $b$.

# Parameter Estimation with Excel Solver - Results

| | | | | | |
|---|---|---|---|---|---|
| Model h'(t) = -b Sqrt(g h(t)) | | | | | |
| | | | | | |
| Model h(t) = (-b Sqrt(g)/2 + h(0)^(1/2))^2 | | | | | |
| | | | | b= | 0.002581 |
| | | | | | |
| | **Zeroed** | **Actual** | **Model** | **SSE** | |
| **Time (s)** | **Time** | **Height (cm)** | | | |
| 8.679 | 0 | 11.1 | 11.09995836 | 1.73426E-09 | |
| 10.866 | 2.187 | 10.6 | 10.51908638 | 0.006547014 | |
| 15.612 | 6.933 | 9.3 | 9.312232219 | 0.000149627 | |
| 18.396 | 9.717 | 8.6 | 8.638501405 | 0.001482358 | |
| 25.781 | 17.102 | 7 | 6.973871293 | 0.000682709 | |
| 31.647 | 22.968 | 5.75 | 5.778477118 | 0.000810946 | |
| 39.282 | 30.603 | 4.4 | 4.390799244 | 8.46539E-05 | |
| 48.182 | 39.503 | 3 | 3.013347567 | 0.000178158 | |
| 56.342 | 47.663 | 2 | 1.977592125 | 0.000502113 | |
| | | | Total SEE | 0.010437581 | |



**CONGRATULATIONS!!!**

We went from

- seeing and collecting data,
- to conjecturing empirical models,
- to building an analytical model from first principles,
- to realizing a differential equation model,
- to solving of the differential equation,
- to estimating our parameter,
- to comparing our model with the actual data.