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STUDENT VERSION
Simulating Phosphorus Levels in the Great Lakes Using Compartmental modeling

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1 SCENARIO DESCRIPTION

The Great Lakes consist of Lakes Superior, Michigan, Huron, Erie, and Ontario and forms the largest system of freshwater lakes by total area in the whole world, see Figure 1. It plays a crucial role in the US economy, serves as a habitat for a diverse range of species of plants and animals, and offers numerous recreational opportunities.

Figure 1: The Great Lakes is a network of deep freshwater lakes in North America.

According to [1], as of 2019, phosphorus levels were too high in the offshore waters of Lake Erie, too low in the offshore waters of Lakes Ontario and Huron, and at the level they should be in the offshore waters of Lake

Superior. Monitoring phosphorus levels in lake systems is important for maintaining water quality, sustaining ecological balance, and safeguarding human health. When the phosphorus level is too high, the water becomes murky and toxic and hence is unsafe for human consumption and enjoyment. When the phosphorus level is too low, plant growth becomes limited, oxygen production is depleted, and the ecosystem becomes imbalanced.

Common ways in which human activities can significantly contribute to the increase of phosphorus levels include agricultural and urban runoff, industrial discharge, and deforestation. Although phosphorus input may be reduced by responsible and intentional agricultural, industrial, and human practices, a biological recycling of phosphorus happens within a lake due to the physical properties of the lake and the biological activities between the plant and animal species in the lake. On the other hand, a natural cause of decrease in phosphorus levels is the natural aging of lake systems.

In this teaching module, we use a highly simplified linear compartmental model to analyze the long-term phosphorus levels in the Great Lakes. In general, phosphorus levels are affected by various geophysical, ecological, and human activities; here we simulate phosphorus levels using the salt-tank compartmental modeling that students encounter in a first-course in Ordinary Differential Equations (ODE).

2 STUDENT ACTIVITIES

2.1 Data Collection

In order to adapt the ideas and techniques that students learned in salt-tank modeling, the students have to collect the following vital information: physical characteristics of the lakes (volumes and retention times) and the geometric network of the system. These data points can be collected by a simple internet search, for example, [2]. Data collected will be used in the next section to compute flow rates along lake connections and to set up the system of ODEs that measures phosphorus levels in the lakes.

Volume and Retention Time of the Lakes

The data points are summarized in Table 1:

Lake	**Volume (in cubic miles)**	**Retention Time (in years)**
Superior	$V_S = 2900$	$RT_S = 191$
Michigan	$V_M = 1180$	$RT_M = 62$
Huron	$V_H = 849$	$RT_H = 21$
Erie	$V_E = 116$	$RT_E = 2.7$
Ontario	$V_O = 393$	$RT_O = 6$

Table 1: Volumes and Retention Times of the Great Lakes.

The retention time or residence time of a lake refers to the average amount of time it takes for water to leave the lake and it is a key parameter in limnology, which is the study of inland waters. Like the volumes of

lakes, the data points above can be directly obtained from the Internet. For example, a retention time of 191 years for Lake Superior would mean that if we put a dye in Lake Superior, it would take about 191 years for the dye to be completely flushed out of the lake through its outflows. Among the five lakes, Superior has the longest retention time because it is large, deep, and has a substantial volume of water.

Geometric Network

The next step is to analyze the geometry of the Great Lakes by looking at a simplified network that connects the lakes. In this module, we assume a geometry as seen in Figure 2, where each lake is represented by a rectangle and lakes are connected with each other by a black arrow. According to [2], Superior's main outlet is the St. Mary's River to Huron; Michigan's main outlet is the Straits of Mackinac to Huron; Huron's main outlet is St. Claire River to Erie; Erie's main outlets are Niagara River and Welland Canal (here, we will assume that it connects to Ontario); and Ontario's main outlet is the St. Lawrence River to the Atlantic Ocean.

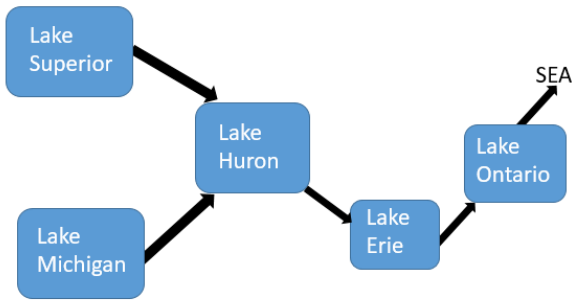

Figure 2: The model assumes a simplified network between the five lakes.

2.2 Flow Rate Computations

Assuming the geometry above, the next step is to compute the flow rate of these connections. We will use the following notation and formula:

$$R_{ij} = \text{Flow Rate from Lake } i \text{ to Lake} j = \frac{\text{Volume of Lake } i}{|\text{Retention Time of Lake } i - \text{Retention Time of Lake } j|}. \quad (1)$$

For example, Superior's flow rate into Huron is

$$R_{SH} = \frac{V_S}{|RT_S - RT_H|} = \frac{2900}{|191 - 21|} = 17.0588 \text{ cubic miles per year}$$

To compute Ontario's flow rate into the Atlantic Ocean, we compute this as volume of Ontario divided by its retention time. The results are summarized in Table 2.

Flow Direction	Flow Rate (in cubic miles/year)
Superior to Huron	R_{SH}
Michigan to Huron	R_{MH}
Huron to Erie	R_{HE}
Erie to Ontario	R_{EO}
Ontario to Atlantic Ocean	R_{OO}

Table 2: Flow Rates of the Great Lakes.

2.3 Equations and General Solution

Let $u_S(t), u_M(t), u_H(t), u_E(t), u_O(t)$ be the amount of phosphorus at some time t in Lakes Superior, Michigan, Huron, Erie, and Ontario, respectively. Using the geometry as a guide, the system of first-order ODEs are

$$\begin{cases} \frac{du_S}{dt} &= -R_{SH}\frac{u_S}{V_S} \\ \frac{du_M}{dt} &= -R_{MH}\frac{u_M}{V_M} \\ \frac{du_H}{dt} &= R_{SH}\frac{u_S}{V_S} + R_{MH}\frac{u_M}{V_M} - R_{HE}\frac{u_H}{V_H} \\ \frac{du_E}{dt} &= R_{HE}\frac{u_H}{V_H} - R_{EO}\frac{u_E}{V_E} \\ \frac{du_o}{dt} &= R_{EO}\frac{u_E}{V_E} - R_{OO}\frac{u_O}{V_O}. \end{cases} \tag{2}$$

When written in matrix notation, we have

$$\begin{pmatrix} u_S' \\ u_M' \\ u_H' \\ u_E' \\ u_O' \end{pmatrix} = \begin{pmatrix} \frac{-R_{SH}}{V_S} & 0 & 0 & 0 & 0 \\ 0 & \frac{-R_{MH}}{V_M} & 0 & 0 & 0 \\ \frac{R_{SH}}{V_S} & \frac{R_{MH}}{V_M} & \frac{-R_{HE}}{V_H} & 0 & 0 \\ 0 & 0 & \frac{-R_{HE}}{V_H} & \frac{-R_{EO}}{V_E} & 0 \\ 0 & 0 & 0 & \frac{R_{EO}}{V_E} & \frac{-R_{OO}}{V_O} \end{pmatrix} \begin{pmatrix} u_S \\ u_M \\ u_H \\ u_E \\ u_O \end{pmatrix} \tag{3}$$

The general solution can be immediately computed; observe that the eigenvalues of the matrix are the diagonal entries

$$\frac{-R_{SH}}{V_S}, \frac{-R_{MH}}{V_M}, \frac{-R_{HE}}{V_H}, \frac{-R_{EO}}{V_E}, \frac{-R_{OO}}{V_O}.$$

These are all negative numbers. This implies that the trivial equilibrium, the phosphorus-free state $(0, 0, 0, 0, 0)$, is stable. As discussed in the introduction, low or zero phosphorus levels are not physically desirable. Due to the simplifications we have assumed however, the phosphorus-free state turns out to be a desirable stable equilibrium.

To write down the general solution, we need to compute the eigenvalues and eigenvectors of the above matrix of coefficients. This can be accomplished via MATLAB:

Lake	Total Phosphorus, 2013 (in μg per liter)	Total Phosphorus, 2013 (in lbs per cubic mile)
Superior	2.10	19297.5
Huron	4.23	38870.6
Erie	18.19	167152.78
Ontario	5.99	55043.71
Michigan*	3.165	29084.033

Table 3: Phosphorus Levels in the Great Lakes.

```
Vs = 2900; Vm = 1180; Vh = 849; Ve = 116; Vo = 393;
Rsh = 17.0588; Rmh = 28.7805; Rhe = 46.3934; Reo = 35.1515; Roo = 65.5;

A = sym([-Rsh/Vs 0 0 0 0; 0 -Rmh/Vm 0 0 0; Rsh/Vs Rmh/Vm -Rhe/Vh 0 0; ...
    0 0 Rhe/Vh -Reo/Ve 0; 0 0 0 Reo/Ve -Roo/Vo])
[EigVec, EigVal] = eig(A)
```

Using the MATLAB results on the eigenvalues and eigenvectors, the general solution is given by

$$\begin{pmatrix} u_S(t) \\ u_M(t) \\ u_H(t) \\ u_E(t) \\ u_O(t) \end{pmatrix} = C_1 e^{-0.1667t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{-0.3030t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -0.45 \\ 1 \end{pmatrix} + C_3 e^{-0.0059t} \begin{pmatrix} 23.9175 \\ 0 \\ 2.8852 \\ 0.5306 \\ 1 \end{pmatrix} + C_4 e^{-0.0244t} \begin{pmatrix} 0 \\ -2.9697 \\ 2.3941 \\ 0.4695 \\ 1 \end{pmatrix} + C_5 e^{-0.0546t} \begin{pmatrix} 0 \\ 0 \\ 1.6803 \\ 0.3697 \\ 1 \end{pmatrix} \tag{4}$$

2.4 Initial Conditions

The next step is to look at the literature and collect data on the historical phosphorus levels in the five lakes. As of this writing, the most recent data according to the Environment and Climate Change Canada is in [3], published in 2015. A summary of the data points that we use in this scenario is in Table 3.

The paper [3] reports that there is no available data for Lake Michigan; in this scenario, we will assume the total phosphorus level in Michigan as the average of the phosphorus levels in Superior and Huron. Also, the paper reports three values for Lake Erie, coming from the West, Central, and East parts of Erie. In

this scenario, we take the average of phosphorus levels in these three areas. Finally, we convert units from micrograms of phosphorus per liter to pounds of phosphorus per cubic mile for unit consistency.

2.5 Simulations

We can now compute the specific solution using the values for the phosphorus levels in 2013. MATLAB's ode45 can plot the solutions readily over the time interval [0, 17] and approximate the phosphorus levels u_S, u_M, u_H, u_E, u_O at the time $t = 17$ (the year 2030.) The MATLAB code is given below with output graphics shown in Figure 3:

```
IC2013 = [19297.5 29084.033 38870.6 167152.78 55043.71];
[tSol, USol] = ode45(@odeGreatLakesPh, [0,17], IC2013);

hold on
plot(tSol, USol(:, 1)); plot(tSol, USol(:, 2)); plot(tSol, USol(:, 3));
plot(tSol, USol(:, 4)); plot(tSol, USol(:, 5));
legend('uS(t)', 'uM(t)', 'uH(t)', 'uE(t)', 'uO(t)')
xlabel('Time'); ylabel('Phosphorus Levels')

%Section Break***************

function dUdt = odeGreatLakesPh(t, U)
uS = U(1); uM = U(2); uH = U(3); uE = U(4); uO = U(5);

Vs = 2900; Vm = 1180; Vh = 849; Ve = 116; Vo = 393;
Rsh = 17.0588; Rmh = 28.7805; Rhe = 46.3934; Reo = 35.1515; Roo = 65.5;

duSdt = -(Rsh/Vs)*uS;
duMdt = -(Rmh/Vm)*uM;
duHdt = (Rsh/Vs)*uS + (Rmh/Vm)*uM - (Rhe/Vh)*uH;
duEdt = (Rhe/Vh)*uH - (Reo/Ve)*uE;
duOdt = (Reo/Ve)*uE - (Roo/Vo)*uO;
dUdt = [duSdt; duMdt; duHdt; duEdt; duOdt];
end
```

In particular, the value of the C parameters in (4) are given by

$$(C_1, C_2, C_3, C_4, C_5) = (390530,\ -353880,\ 806.8345,\ 9793.5,\ 7793.7).$$

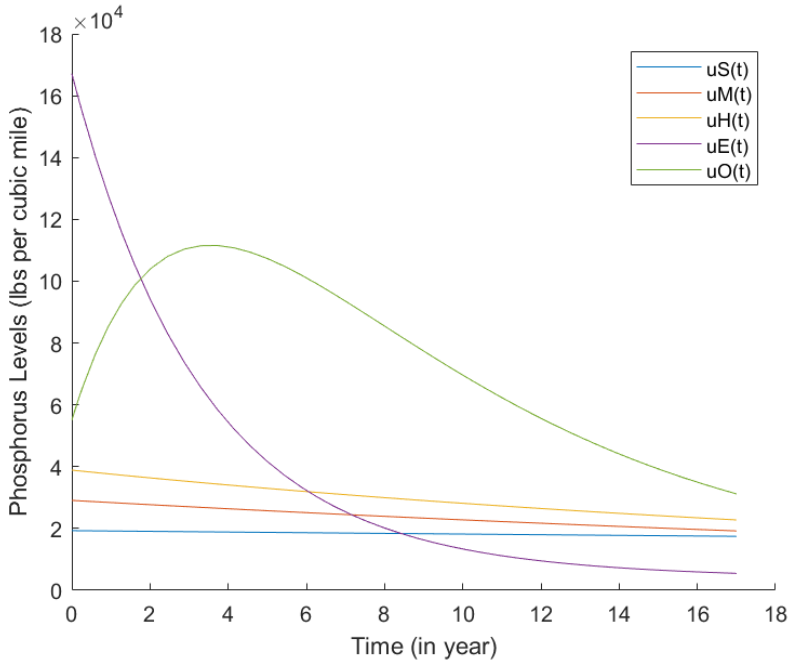

Figure 3: Simulation of Phosphorus Levels using the model (2) and plotted by MATLAB via ode45.

3 NOTES FOR THE TEACHER

Here, we list some possible activities for the students:

1. Using data in Table 1 and the flow rate formula (1), compute all flow rates in Table 2. Compare results with a similar example found in [4], page 547.
2. For each lake, compute the times at which its phosphorus level reaches a) 25 percent, b) 50 percent, and c) 75 percent of its initial value.
3. There are US and Canadian government efforts to control the phosphorus levels in the Great Lakes. Conduct research on such efforts and investigate their effectiveness using ODEs.
   (a) Modify the equations in (2) to include clean-up efforts. Students can investigate phosphorus levels using various theoretical clean-up assumptions, for example, using one or a combination of:
      i. constant
      ii. linear
      iii. periodic
      iv. impulsive (use Dirac delta), and
      v. discontinuous (use Heaviside).
   (b) Using data on clean-up efforts, validate and test simulation results.
4. Georgian Bay is a large bay of Lake Huron and is located in the borders of Ontario, Canada, Figure 4.

Figure 4: Georgian Bay in Canada, image from https://thenarwhal.ca/ontario-great-lakes-bill-23/.

(a) Modify the equations (2) to include the Georgian Bay in the geometric network of the bodies of water.

(b) Conduct research on the following facts about the Georgian Bay: volume, retention time, and phosphorus levels.

(c) Simulate the Phosphorus levels at the Georgian Bay by modifying the MATLAB code presented in Section 2.5.

5. In 2020, Environment and Climate Change Canada released a report on the Phosphorus levels in the Great Lakes. A summary of the results is pictured in Figure 5. Modify the MATLAB code in Section 2.5 to include Georgian Bay and remove Lake Michigan. Use data given in [1] to predict Phosphorus status in the offshore waters of the Great Lakes in 2030, 2040, and 2050. When updating the code, be mindful of the scales and units of measure.

References

1. Environment and Climate Change Canada. 2020, *Canadian Environmental Sustainability Indicators: Phosphorus Levels in the Offshore Waters of the Great Lakes*. Available at https://www.canada.ca/en/environment-climate-change/services/environmental-indicators/phosphorus-levels-off-shore-great-lakes.html.

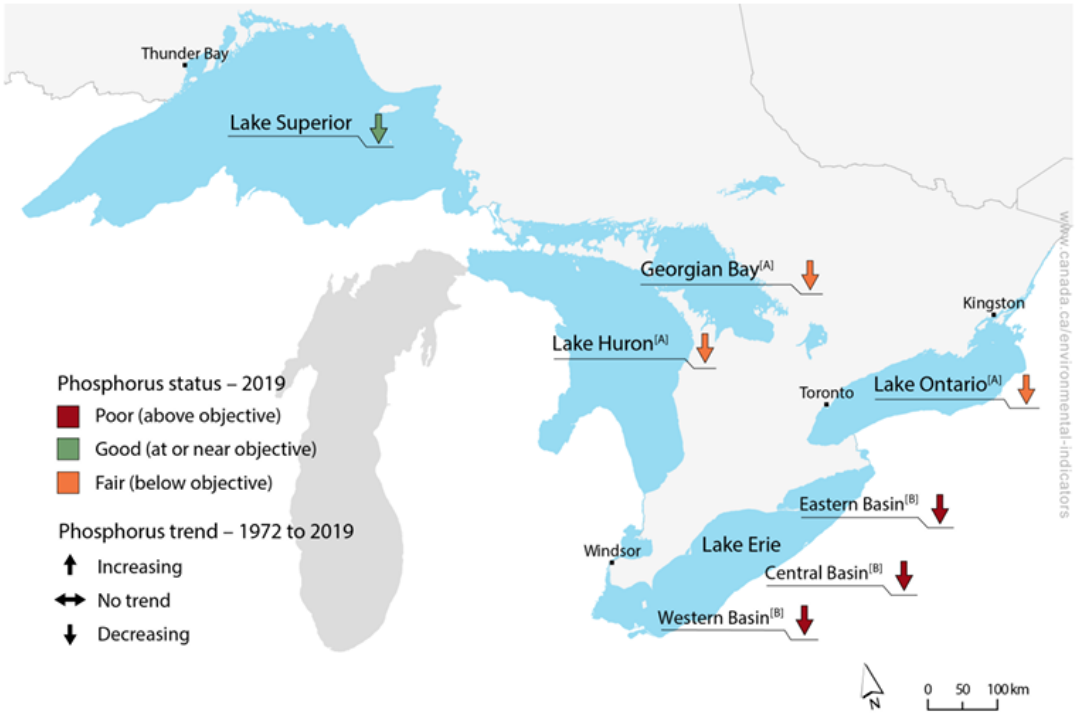

Figure 5: Status and trends of phosphorus levels in the offshore waters of the Great Lakes, 1972-2019, reproduced from https://www.canada.ca/en/environment-climate-change/services/environmental-indicators/phosphorus-levels-off-shore-great-lakes.html.

2. Michigan Sea Grant (Michigan State University, University of Michigan, and the National Oceanic and Atmospheric Administration)/. 2023. *Great Lakes Fast Facts*. Available at https://www.michiganseagrant.org/topics/great-lakes-fast-facts/.

3. Dove, Alice and Steven C. Chapra. 2015. *Long-term trends of nutrients and trophic response variables for the Great Lakes Limnol. Oceanogr*. 60: 696-721, Association for the Sciences of Limnology and Oceanography, doi:10.1002/lno.10055. Available at https://aslopubs.onlinelibrary.wiley.com/doi/10.1002/lno.10055.

4. MacCluer, Barbara D., Paul S. Bourdon, and Thomas, L. Kriete. 2019. *Differential Equations: Techniques, Theory, and Applications*.American Mathematical Society, Providence RI USA.