

# Math for Biology Workbook

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This workbook is designed for MATH 119, the Mathematics for Biology course at Radford University.

It is expected that the students have a working knowledge of basic mathematics at a high school level, but calculus, pre-calculus or trigonometry are not required. The topics have been selected based on their relevance for biology. Logarithms and exponential are extremely useful in many areas of biology, and probability and the binomial distribution are included for their use in genetics.

Difference equations are included because they can be easily implemented in EXCEL or other computing environments. This opens the door to first simulations of population growth without the Calculus machinery. Of course Calculus or Linear Algebra are inevitable once stability issues arise. Chapter 16 looks at first order difference equations; another chapter with examples of higher order difference equations (Fibonacci numbers, Leslie population models) and systems of difference equations (SIR, predator-prey) is planned. Examples, mind you, not a detailed analysis, which would be well beyond the scope of this workbook.

This workbook is not designed like an ordinary math book where the concepts build logically on top of each other, and one chapter is a prerequisite for the next. For instance, most students know how to cancel common factors, expand an expression or solve simple equations; and we use them from the onset, while a detailed discussion of algebra operations or strategies for solving an equation factoring will be presented much later.

Calculations are usually carried out with highest possible precision, but intermediate results (to help the student follow the steps) and final answers are rounded to a few decimal places, because most of the time two or three significant digits matter.

This is not a recipe book (as in "take the square root, double it, and add three to the answer"). We emphasize the thoughts and ideas which go into a particular concept, along with pitfalls and common errors. In many cases we will present different methods to solve a problem, and in the end the student has to select the approach with which he or she is most comfortable.

On occasion the workbook goes past the bare minimum of the course. These parts could be considered further reading material for the interested student, or they may serve as reference material.

In its first run the MATH 119 course was taught using the text "Quantitative Reasoning and the Environment" by Langkamp and Hull. Some of their ideas, examples and exercises have been incorporated into this workbook.

The graphs were produced with EXCEL, Maple and Paint.

Many thanks to the biology faculty who have let me audit their classes, and to those who have contributed teaching material, and to those who served as my sounding board for this endeavor.

# 1 Scientific Notation

In biology, we often have to deal with very large or very small quantities. For instance, the adult human body consists of approximately 37 trillion cells; rod shaped bacteria, bacilli, are between 1 to 4 micrometers (microns) long; drinking water may contain up to 10 parts per billion of arsenic by EPA standards, and so on. Scientific notation is a convenient way to express such numbers.

## 1.1 Scientific Notation

**Definition (Scientific Notation):** Every positive real number  $x > 0$  can be written in the form

$$x = a 10^n$$

where the exponent  $n$  is an integer (whole number). The number  $1 \leq a < 10$  is called the *mantissa*, and the number  $n$  indicates the order of *magnitude*.

**Examples:**

$$\begin{aligned} 256 &= 2.56 10^2 \\ 0.0035 &= 3.5 10^{-3} \\ 6.9 &= 6.9 10^0 \end{aligned}$$

The requirement that  $1 \leq a < 10$  ensures that scientific notation is unique. Otherwise we could have  $22 = 22 10^0 = 2.2 10^1 = 0.22 10^2$ . For negative numbers we can just include a negative sign, as in  $-414 = -4.14 10^2$ . If you want to write  $x = 0$  in scientific notation, you have to allow for  $a = 0$ .

## 1.2 Significant Figures

**Significant Digits (Figures):** This is the number of digits of the mantissa.

**Examples:**

$$\begin{aligned} 256 &= 2.56 10^2 & 3 \text{ significant digits} \\ -0.45 &= -4.5 10^{-1} & 2 \text{ significant digits} \end{aligned}$$

Usually you can determine the significant digits by counting the digits left to right, ignoring leading zeros.

Trailing zeros are a little tricky. Undoubtedly,

$$256.0 = 256$$

The first number contains four significant digits, the second only three. In scientific notation we would write  $2.560 10^2$ , and  $2.56 10^2$ , respectively. The first mantissa contains four digits, the second only three.

Without scientific notation there remains a bit of ambiguity when dealing with large numbers. For example, it is not possible to tell how many significant digits the number

$$42,600,000,000$$

contains. We have

$$42,600,000,000 = 4.26 10^9 = 4.260 10^9$$

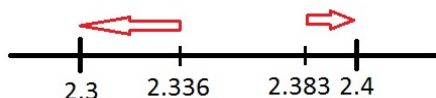
The number has at least three significant figures, but it could be more.

As we work applied problems, we should keep the number of significant figures about the same. For instance, in  $\frac{423}{22.7}$ , both, numerator and denominator have three significant digits. It makes no sense to write  $\frac{423}{22.7} = 18.63436123$ . This is pretending that we have very high accuracy, and that we stand behind the correctness of all of these ten digits. Instead we should round to three places and write

$$\frac{423}{22.7} = 18.6$$

### 1.3 Rounding

**Rounding:** Always round to the nearest.



#### Examples:

Round to two significant figures:

$$2.336 \approx 2.3$$

$$2.383 \approx 2.4$$

Round to three significant figures:

$$-22.469 \approx -22.5$$

$$\frac{8}{3} \approx 2.67$$

When rounding to the nearest, the cutoff is five. Digits in the range 0 – 4 are rounded down, digits in the range 5 – 9 are rounded up. If the digit in question is exactly equal to five, we need a tie breaker. For instance, the number 2.35 is exactly in the middle between 2.3 and 2.4. A recommended strategy is to *round to even*. Example:

$$2.35 \approx 2.4$$

$$2.45 \approx 2.4$$

$$2.55 \approx 2.6$$

$$2.65 \approx 2.6$$

We rounded to 2.4 or 2.6, rather than to 2.5.

In a more complicated calculation which involves several steps, it is a good idea to keep the highest precision possible throughout, and wait with rounding until the final answer is calculated.

For simplicity's sake we shall use "=" from here on, even when we are rounding, as in  $\pi = 3.142$ , or  $\frac{10}{7} = 1.4286$ . Use common sense when deciding the number of significant figures!

### 1.4 Unit Prefixes

The major prefixes are listed in the Figure 1.

tera	T	one trillion	$10^{12}$
giga	G	one billion	$10^9$
mega	M	one million	$10^6$
kilo	k	one thousand	$10^3$
hecto	h	one hundred	$10^2$
deca	da	ten	$10^1$
		one	$10^0$
deci	d	one tenth	$10^{-1}$
centi	c	one hundredth	$10^{-2}$
milli	m	one thousandth	$10^{-3}$
micro	$\mu$	one millionth	$10^{-6}$
nano	n	one billionth	$10^{-9}$
pico	p	one trillionth	$10^{-12}$

Figure 1: Unit Prefixes

As we go through the table, the powers of ten change in increments of three. This is similar to the way we group digits in blocks of three. Example:

$$\begin{aligned} &23,400,000,000,000 \\ &= 23.4 \cdot 10^{12} = 23.4 \text{ trillion} \end{aligned}$$

There is also a fine tuning around unity. This accommodates things like centimeter, or hectoliter, et cetera.

**CAUTION:** A billion may not equal a billion! The *short scale*, where a billion equals  $10^9$ , is used in the U.S., Great Britain and in the Arabic world. Continental Europe and Latin America use the *long scale*, where a billion equals  $10^{12}$ , and  $10^9$  is called a milliard in that number scale.

### 1.5 Calculator

The descriptions are given for TI calculators. The labels of the keys may be different for other brands, but the ideas are the same.

Use the EE key to enter numbers given in scientific notation. For example

$$2.34 \cdot 10^{-5} \quad \text{becomes} \quad 2.34EE - 5$$

Calculators will switch to scientific notation when numbers become too large or too small. On the TI-84+ the number 0.000234 is shown as

$$2.34E - 4$$

which is to be interpreted as  $2.34 \cdot 10^{-4}$ .

You can also change the MODE to SCI. In this case all numbers will be displayed in scientific notation. For instance

$$25,000 \quad \text{becomes} \quad 2.5E4$$

The engineering mode ENG is very useful. Here the powers of ten are incremented in steps of three, and we can use the unit prefix table directly. For example,

$$25,000,000 \quad \text{becomes} \quad 25E6$$

and the number is 25 million.

## 1.6 Worked Problems

- Convert the given numbers to scientific notation

- 8,630,000
- 0.004,56
- 63.9 trillion
- $0.25 \cdot 10^{11}$

Solutions:

- $8,630,000 = 8.63 \cdot 10^6$   
Move the decimal point to the left by six spaces.
- $0.004,56 = 4.56 \cdot 10^{-3}$   
Move the decimal point to the right by three spaces.

- From the table we obtain that a trillion equals  $10^9$ , and thus

$$\begin{aligned} 63.9 \text{ trillion} &= 63.9 \cdot 10^9 \\ &= 6.39 \cdot 10^{10} \end{aligned}$$

The last step was necessary because 63.9 is not between 1 and 10.

$$(d) \quad 0.25 \cdot 10^{11} = 2.5 \cdot 10^{10}$$

- Round the numbers to four significant figures.

- 345,678
- 0.0691
- 22.715
- $\pi$
- $\frac{22}{7}$

Solutions:

- $345,678 \approx 345,700$   
678 is rounded to 700.
- $0.0691 \approx 0.06910$   
This problem is questionable, as the original number only has three significant figures.
- $22.715 \approx 22.72$   
Round to even (2), rather than down to 22.71.
- $\pi = 3.141592\dots \approx 3.142$
- $\frac{22}{7} = 3.14285\dots \approx 3.143$

- Compute the expression and express the result in scientific notation.

- $3.521 \cdot 10^7 - 2.63 \cdot 10^5$
- $6.9 \cdot 10^{-4} \times 2.34 \cdot 10^6$

Solutions: Calculators will solve these problems just fine. Here we give some detail.

(a)  $3.521 \cdot 10^7 - 2.63 \cdot 10^5$

We cannot subtract 2.63 from 3.521 directly, because the exponents are different. Instead we change the last term so that the exponent becomes 7 as well.

$$\begin{aligned} & 3.521 \cdot 10^7 - 2.63 \cdot 10^5 \\ = & 3.521 \cdot 10^7 - 0.0263 \cdot 10^7 \\ = & (3.521 - 0.0263) \cdot 10^7 \\ = & 3.4947 \cdot 10^7 \\ \approx & 3.495 \cdot 10^7 \end{aligned}$$

The larger number had just four significant figures, and we shouldn't put any faith in the fifth digit.

(b)  $6.9 \cdot 10^{-4} \times 2.34 \cdot 10^6$

This is straight multiplication. Multiply mantissas and powers of ten separately:

$$\begin{aligned} & 6.9 \cdot 10^{-4} \times 2.34 \cdot 10^6 \\ = & 6.9 \times 10^{-4} \times 2.34 \times 10^6 \\ = & 16.146 \times 10^2 \approx 1.6 \cdot 10^3 \end{aligned}$$

## 1.7 Exercises

1. Convert to scientific notation:

- (a) 43,000,000,000
- (b) 0.000,04
- (c) 12
- (d) 6
- (e)  $\frac{1}{4}$
- (f) 75.6 trillion
- (g) 53 millionth

2. Compute the expressions, and express the results in scientific notation. Round to four significant digits, if necessary.

(a)  $1.8 \cdot 10^7 - 8 \cdot 10^6$

(b)  $6.4 \cdot 10^{-7} \times 3.75 \cdot 10^3$

(c)  $\frac{5.7 \cdot 10^4}{7.7 \cdot 10^{-5}}$

(d)  $\frac{6 \cdot 10^5 + 7.2 \cdot 10^{-2}}{2.5 \cdot 10^2 + 3 \cdot 10^{-5}}$

(e)  $(5 \cdot 10^4)^2$

(f)  $(5 \cdot 10^5)^2$

(g)  $(5 \cdot 10^{-4})^2$

(h)  $\sqrt{4.9 \cdot 10^{-5}}$

(i)  $\sqrt{4.9 \cdot 10^{-4}}$

3. How many seconds are in a year? Use scientific notation and round the result to four significant digits.

## Answers

1. (a)  $4.3 \cdot 10^{10}$

(b)  $4 \cdot 10^{-5}$

(c)  $1.2 \cdot 10^1$

(d)  $6 \cdot 10^0$

(e)  $2.5 \cdot 10^{-1}$

(f)  $7.56 \cdot 10^{13}$

(g)  $5.3 \cdot 10^{-5}$

2. (a)  $10^7$

(b)  $2.4 \cdot 10^{-3}$

(c)  $7.043 \cdot 10^8$

(d)  $2.4 \cdot 10^3$

(e)  $2.5 \cdot 10^9$

(f)  $2.5 \cdot 10^{11}$

(g)  $2.5 \cdot 10^{-7}$

(h)  $7 \cdot 10^{-3}$

(i)  $2.214 \cdot 10^{-2}$

3.  $3.154 \cdot 10^7$  seconds



## 2 Metric System

The metric system, or more specifically SI units (Système International d'Unités), dominates the scientific literature. An adult human is about 1.7 meters tall; the green tree frog (*Hyla cinerea*) is about 4 to 5 centimeters long; the common vampire bat (*Desmodus rotundus*) weighs about 55 to 60 gram; blood sugar levels are measured in milligram per deciliter, to name a few examples.

### 2.1 Basic Physical Units

Most physical units are composed of the three basic units: length (L), time (T) and mass (M). In the metric system these are measured in meters (m), seconds (sec) and gram (g), respectively. Other SI units are Ampere (A) for electric current, Kelvin (K) for temperature, mole (mol) for amount of substance and candela (cd) for luminous intensity.

#### Historical Background:

1. Meter: The distance from the pole to the equator is set as

$$10,000 \text{ km} = 10^7 \text{ m}$$

which makes one meter a ten millionth of that distance.

2. A second is based on the rotation of the Earth. One revolution about its own axis takes 24 hours, which is equivalent to

$$24 \times 60 \times 60 \text{ sec} = 86,400 \text{ sec}$$

3. A kilogram is the mass of one liter (L) of water (a liter is set as a one thousandth of a cubic meter). A gram then is a one thousandth of this quantity.

The definitions have since been updated to obtain higher precision.

**Compound Units** are made up from the basic units by multiplication or division. The most popular terms are summarized in the table.

Area	$\text{length} \times \text{length}$	$L^2$
Volume	$\text{area} \times \text{length}$	$L^3$
Velocity	$\frac{\text{displacement}}{\text{time}}$	$\frac{L}{T}$
Acceleration	$\frac{\text{change of velocity}}{\text{time}}$	$\frac{L}{T^2}$
Flux	$\frac{\text{volume}}{\text{time}}$	$\frac{L^3}{T}$
Density	$\frac{\text{mass}}{\text{volume}}$	$\frac{M}{L^3}$
Force	$\text{mass} \times \text{acceleration}$	$\frac{ML}{T^2}$
Work	$\text{force} \times \text{displacement}$	$\frac{ML^2}{T^2}$
Power	$\frac{\text{work}}{\text{time}}$	$\frac{ML^2}{T^3}$
Pressure	$\frac{\text{force}}{\text{area}}$	$\frac{M}{LT^2}$

Figure 2: Compound Units

### 2.2 Metric Measurements

#### 2.2.1 Length

Length is measured in meters (m). For large distances use kilometers, where

$$1 \text{ km} = 10^3 \text{ m} = 1,000 \text{ m}$$

For small lengths centimeters (cm), millimeters (mm) or microns ( $\mu\text{m}$ ) are commonly used

$$1 \text{ cm} = 10^{-2} \text{ m} = 0.01 \text{ m}$$

$$1 \text{ mm} = 10^{-3} \text{ m} = 0.001 \text{ m}$$

$$1 \text{ }\mu\text{m} = 10^{-6} \text{ m} = 0.000,001 \text{ m}$$

This boils down to knowing your prefixes.

### 2.2.2 Area

The basic unit is the square meter (sq. m or  $\text{m}^2$ ), which is the area of a square where both sides are one meter long.

Large areas are expressed in square kilometers, the area of 1 km by 1 km square. Specifically,

$$\begin{aligned} 1 \text{ km}^2 &= 1 \text{ km} \times 1 \text{ km} \\ &= 1,000 \text{ m} \times 1,000 \text{ m} \\ &= 1,000,000 \text{ m}^2 = 10^6 \text{ m}^2 \end{aligned}$$

Beware that a kilo of square meters ( $1,000 \text{ m}^2$ ) is NOT equivalent to a square kilometer!

A hectare (ha) is the area of a square of size 100 m by 100 m. Therefore

$$\begin{aligned} 1 \text{ ha} &= 100 \text{ m} \times 100 \text{ m} \\ &= 10,000 \text{ m}^2 = 10^4 \text{ m}^2 \\ 1 \text{ km}^2 &= 100 \text{ ha} \end{aligned}$$

Small areas can be expressed in square centimeters or square millimeters:

$$\begin{aligned} 1 \text{ cm}^2 &= 1 \text{ cm} \times 1 \text{ cm} \\ &= 0.01 \text{ m} \times 0.01 \text{ m} \\ &= 0.000,1 \text{ m}^2 = 10^{-4} \text{ m}^2 \\ 1 \text{ mm}^2 &= 1 \text{ mm} \times 1 \text{ mm} \\ &= 0.001 \text{ m} \times 0.001 \text{ m} \\ &= 0.000,001 \text{ m}^2 = 10^{-6} \text{ m}^2 \end{aligned}$$

It follows that

$$\begin{aligned} 1 \text{ m}^2 &= 10,000 \text{ cm}^2 \\ 1 \text{ m}^2 &= 1,000,000 \text{ mm}^2 \\ 1 \text{ cm}^2 &= 100 \text{ mm}^2 \end{aligned}$$

### 2.2.3 Volume

The natural unit for volume is the cubic meter ( $\text{m}^3$ ), which is the volume of a cube whose sides are all one meter long. This unit is rather large, and most useful applications use liters (L), or even smaller quantities such as milliliters (mL).

A liter is defined to the volume of the cube whose sides are 10 cm long. In terms of cubic meters we find ( $10 \text{ cm} = 0.1 \text{ m}$ )

$$\begin{aligned} 1 \text{ L} &= 0.1 \text{ m} \times 0.1 \text{ m} \times 0.1 \text{ m} \\ &= 0.001 \text{ m}^3 \end{aligned}$$

and therefore  $1 \text{ m}^3 = 1,000 \text{ L}$ .

In reference cubic centimeters we see that

$$\begin{aligned} 1 \text{ L} &= 10 \text{ cm} \times 10 \text{ cm} \times 10 \text{ cm} \\ &= 1,000 \text{ cm}^3 \end{aligned}$$

Therefore

$$1 \text{ mL} = 1 \text{ cm}^3$$

that is, a milliliter is the equivalent of a cubic centimeter!

### 2.2.4 Velocity and Acceleration

Velocity is measured in meters per second (m/sec) or in kilometers per hour (kmh or km/h), acceleration is measured in meters per second squared ( $\text{m/sec}^2$ ). This is meters per second per second.

Important velocities:

The speed of sound is 344 m/sec.

The speed of light is 299,792,458 m/sec.

The acceleration due to gravity on Earth is  $9.807 \text{ m/sec}^2$ .

### 2.2.5 Flux

This quantity<sup>1</sup> is the product of area and velocity, or equivalently, the ratio volume per time.

$$\begin{aligned}\text{flux} &= \text{area} \times \text{velocity} = L^2 \times \frac{L}{T} \\ &= \frac{L^3}{T} = \frac{\text{volume}}{\text{time}}\end{aligned}$$

Flux can be measured in liters per second.

**Example:** A small river with cross sectional area of  $20 \text{ m}^2$ , and average velocity of  $2 \text{ m/sec}$  transports

$$\begin{aligned}20 \text{ m}^2 \times 2 \text{ m/sec} &= 40 \frac{\text{m}^3}{\text{sec}} \\ &= 40,000 \frac{\text{L}}{\text{sec}}\end{aligned}$$

### 2.2.6 Mass

The basic unit for mass is a kilogram (kg), which, under the appropriate conditions, equals the mass of one liter of water.

A kilogram can be further partitioned into grams (g) or milligrams (mg), where

$$\begin{aligned}1 \text{ g} &= 0.001 \text{ kg} \\ 1 \text{ mg} &= 0.001 \text{ g}\end{aligned}$$

or equivalently

$$\begin{aligned}1 \text{ kg} &= 1,000 \text{ g} \\ 1 \text{ g} &= 1,000 \text{ mg}\end{aligned}$$

For large masses we use the metric ton (tonne)

$$1 \text{ tonne} = 1,000 \text{ kg}$$

---

<sup>1</sup>In the sequel we will use the term "flux" in the sense of a volumetric flow rate. In general terms, flux is defined as the product of density and velocity. For incompressible fluids (water, blood) these definitions are equivalent.

### 2.2.7 Density

This is mass per volume, and usually density is denoted by  $\rho$ . A common unit is gram per cubic centimeter ( $\text{g/cm}^3$ ).

Notice that

$$\frac{\text{g}}{\text{cm}^3} = \frac{\text{kg}}{\text{L}} = \frac{\text{tonne}}{\text{m}^3}$$

The density of water is  $1 \text{ g/cm}^3 = 1 \text{ kg/L}$ , the density of air is about  $1.2 \text{ mg/L}$  and the density of the Earth is  $5.52 \text{ g/cm}^3$ .

### 2.2.8 Force

Force is the product of mass and acceleration. Its SI unit is Newton (N), which is defined as

$$1 \text{ N} = \frac{1 \text{ kg} \cdot 1 \text{ m}}{1 \text{ sec}^2}$$

A much smaller unit for force is dyne, which is defined as

$$1 \text{ dyne} = \frac{1 \text{ g} \cdot 1 \text{ cm}}{1 \text{ sec}^2}$$

It follows that

$$1 \text{ N} = 100,000 \text{ dyne}$$

The force of 1 kg on Earth is 9.81 N.

### 2.2.9 Work

Work is the product of force and displacement. It is equivalent to energy, and to momentum (mass times the square of the velocity). Common units for work are Joule (J), erg, Watt hours (Wh) or calories. Definitions:

$$\begin{aligned}1 \text{ J} &= 1 \text{ N} \times 1 \text{ m} \text{ (Newton meter)} \\ 1 \text{ erg} &= 1 \text{ dyne} \times 1 \text{ cm} \\ 1 \text{ Wh} &= 3,600 \text{ J} \\ 1 \text{ cal} &= 4.1840 \text{ J}\end{aligned}$$

Joule is considered the official SI unit, the others are listed for convenience.

A calorie (cal) is the energy required to raise the temperature of one gram of water by one degree Celsius.

Beware that the calories listed on food items are kilo calories (kcal), and they are sometimes written as Cal, with an upper case C.

### 2.2.10 Power

Power is the ratio of work per time. The SI unit for power is 1 Watt (W) = 1 Joule/sec.

One horsepower is the equivalent of 745.7 Watt.

While Watt is a measurement of power, Watt hours (Wh) measure energy (power multiplied by time results in work).

**Example:** If you use a 25 Watt light bulb for one hour, you use up 25 Watt hours, and the energy consumption is

$$= \frac{25 \text{ W} \times 1 \text{ h}}{\frac{25 \text{ J}}{\text{sec}}} \times 3,600 \text{ sec} = 90,000 \text{ J}$$

### 2.2.11 Pressure

Pressure is the ratio force per area, and its SI unit is Pascal (Pa), defined as

$$1 \text{ Pa} = \frac{1 \text{ N}}{\text{m}^2}$$

In the medical field blood pressure is measured in mmHg (millimeter of mercury) and 1 mmHg = 133 Pa.

The tire pressure in your car should be about 32 PSI = 221,000 Pa = 221 kPa, and the atmospheric pressure at sea level is about 101 kPa.

### 2.2.12 Temperature

Biologists usually measure temperature<sup>2</sup> in degrees Celsius. It was originally set so that water freezes at 0°C and it boils at 100°C. The relation to Kelvin (K) is that

$$\text{K} = ^\circ\text{C} + 273.15$$

## 2.3 Worked Problems

1. (a) How many millimeters make up a kilometer?
- (b) Convert 0.000,000,044 km to meters and microns.

Solutions:

- (a) 1 km = 1,000 m = 1,000,000 mm
- (b) 0.000,000,044 km = 0.000,044 m  
= 44  $10^{-6}$  m = 44  $\mu\text{m}$

2. (a) Convert 25,000 square meters to hectare.
- (b) Is it reasonable that a leaf has area 30 million square millimeters? What about 30 thousand square centimeters?
- (c) How many cubic millimeters make up a milliliter?

Solutions:

- (a) 25,000 m<sup>2</sup> = 2.5  $\times$  10,000 m<sup>2</sup>  
= 2.5 ha
- (b) 30 million square millimeters is the area of a rectangle with sides 6,000 mm (= 6 m) and 5,000 mm (5 m). This is unreasonably large for a leaf. 30,000 mm<sup>2</sup> = 150 mm  $\times$  200 mm, which is a 15 cm by 20 cm rectangle, a reasonable size for a big leaf.

---

<sup>2</sup>The official SI unit for temperature is Kelvin.

Yes, leaves are not rectangular, but we are just comparing sizes of leaves to sizes of rectangles.

- (c) 1 mm = 0.1 cm. Therefore

$$\begin{aligned} 1 \text{ mm}^3 &= (0.1 \text{ cm})^3 \\ &= 0.001 \text{ cm}^3 = 0.001 \text{ mL} \end{aligned}$$

3. What is the flux through a blood vessel whose cross sectional area is  $4 \text{ mm}^2$  when the blood passes through the vessel at  $7 \text{ mm/sec}$ ?

Solution:

$$\begin{aligned} &4 \text{ mm}^2 \times 7 \text{ mm/sec} \\ &= 28 \frac{\text{mm}^3}{\text{sec}} = 0.028 \frac{\text{mL}}{\text{sec}} \end{aligned}$$

and it will take about 36 seconds for one milliliter of blood to pass through the vessel.

4. (a) What is the mass of 12 mL of water?  
(b) What is the mass of one cubic meter of water?

Solutions:

- (a) The mass of one liter of water is 1 kg = 1,000 g. Multiply by 0.012 to obtain 12 mL, and the mass is 12 g.  
(b) The mass of one liter of water is 1 kg = 1,000 g. Multiply by 1,000 to obtain a cubic meter; the resulting mass is 1,000 kg = 1 tonne.

5. How many tonnes are in

- (a) 23 kg,  
(b) 5,709,000,000 mg?

Solutions:

- (a) 23 kg = 0.023 tonnes

$$\begin{aligned} \text{(b)} \quad 5,709,000,000 \text{ mg} &= 5,709,000 \text{ g} \\ &= 5,709 \text{ kg} = 5.709 \text{ tonnes} \end{aligned}$$

6. What work is required to lift 75 kg by 5 cm?

Solution:

The force exerted by 75 kg on Earth is

$$F = 75 \text{ kg} \times 9.81 \frac{\text{m}}{\text{sec}^2} = 736 \text{ N}$$

Hence, the resulting work is

$$\begin{aligned} W &= 736 \text{ N} \times 5 \text{ cm} \\ &= 736 \text{ N} \times 0.05 \text{ m} = 36.8 \text{ J} \end{aligned}$$

7. A bird eats a worm containing 150 J of energy. How many kcals (food) is that?

Solution: 1 cal = 4.1840 J. Therefore

$$\begin{aligned} 150 \text{ J} &= \frac{150}{4.1840} \text{ cal} \\ &= 35.85 \text{ cal} \approx 0.036 \text{ kcal} \end{aligned}$$

8. (a) What physical quantity is obtained when you divide velocity by acceleration?  
(b) What physical quantity is obtained when you multiply mass and the square of a velocity?

Solutions:

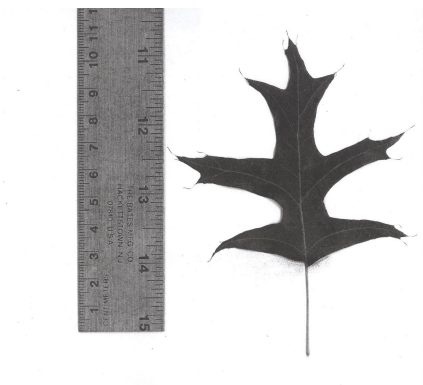
$$\text{(a)} \quad \frac{\text{velocity}}{\text{acceleration}} = \frac{L/T}{L/T^2} = T$$

The answer is time.

$$\begin{aligned} \text{(b)} \quad \text{mass} \times \text{velocity}^2 &= M \times \left(\frac{L}{T}\right)^2 \\ &= \frac{ML^2}{T^2} = \text{work} \end{aligned}$$

## 2.4 Exercises

- Convert 2,000 mm to cm, to m and to km.
  - Convert 2.12 cm to meters.
  - Convert 0.000,021 m to  $\mu\text{m}$ .
- A fish measures 37.5 cm in length. Convert the length to mm and to m.
- Convert to meters and express the results in scientific notation. Some values are averages or crude approximations.
  - 53 pm, the atomic radius of hydrogen,
  - 600 nm, the wavelength of visible light,
  - 2  $\mu\text{m}$ , size of bacteria,
  - 100  $\mu\text{m}$ , the thickness of human hair,
  - 0.76 mm, the thickness of ID cards,
  - 3 cm, the size of an acorn,
  - 1.75 m, the height of a human,
  - 42.2 km, the length of a marathon,
  - 6,371 km, the radius of the Earth,
  - 384,000 km, the distance to the moon, and
  - 150 million km, the distance to the sun.
- How many kilometers are covered by a 4 x 400m relay?
- A particle moves 14 cm, and then 9 km, and then 455  $\mu\text{m}$ , and finally 87mm. Express the total distance in meters. Do not round the result.
- Hesperoyucca whipplei is among the fastest growing plants on Earth, and it was recorded to have grown 3.7m in 14 days. What is the growth rate in mm per hour?
- A base pair of DNA is approximately 0.340 nm long (1 nm = 1 nanometer = 0.000,000,001 m). The human genome is about 3.2 billion base pairs long. What is the length of human DNA measured in meters?
- How many square kilometers are the equivalent of 4,000 hectares?
- How many liters does a rectangular tank with dimension 40 cm x 50 cm x 80 cm hold?
- How many milliliter of water fit into a container of size 25mm x 25mm x 10mm?
- A leaf has area 25 cm<sup>2</sup>. Convert the area to square millimeters.
- How many cubic millimeters fit into a solid with volume 0.03 L?
- How many cubic millimeters are contained in one milliliter?
- A roll of bandage is 10cm wide and 2.5m long. What is its area?
- How many 2.5 mL doses of a vaccine can be obtained from 1.5 L?
- How many liters of water fell on 5 hectares of land when the rain was measured at 4cm?
- Estimate the area of the leaf pictured in the figure.



18. The cross sectional area of a small stream is  $0.25 \text{ m}^2$ , and it flows at  $0.2 \text{ m/sec}$ . How much water passes through the creek each second?
19. True/False: The mass of one milliliter of water is one gram.
20. How many grams are the equivalent of  $0.045 \text{ tonnes}$ ?
21. How much do 4 cubic centimeters of water weigh?
22. An African elephant weighed in at  $6.8 \text{ tonnes}$ . Convert the weight to kilogram and to gram.
23. A liver cell has mass  $0.000,000,003,5 \text{ g}$ .
  - (a) Convert this mass to mg and  $\mu\text{g}$ .
  - (b) How many cells make up one gram of liver tissue?
24. What (compound) unit is obtained when force is divided by pressure?
25. What (compound) unit is obtained when force is multiplied by velocity?
26. What physical unit could be measured in
  - (a) kilogram per liter?
  - (b) millimeter per day?
  - (c) cm per minute squared ( $\text{cm/sec}^2$ )?
  - (d) kg per meter per hour squared ( $\frac{\text{kg}}{\text{m hr}^2}$ )?
27. (a) What work is required to raise 55 kg by 6 m?  
 (b) How much power is required to raise 55 kg by 6 m in 12 seconds?
28. Calculate how much energy in Joules and in megajoules your body has effectively released into the environment if you go for a run and burn 600 Calories. (Remember Calorie  $\neq$  calorie)
29. A cubic meter of gold has a mass of 19.3 tonnes. What is the density of gold in  $\text{g/cm}^3$  (gram per cubic centimeter)?
30. An engineering website records the density of diamonds as  $3,500 \text{ kg/m}^3$ .
  - (a) Convert this value to  $\text{g/cm}^3$ .
  - (b) What is the volume of a 7 carat diamond? Use:  $1 \text{ carat} = 200 \text{ mg}$ .
31. What is the pressure exerted by 100 dyne on  $5 \text{ cm}^2$ ?
32. Gasoline has a density of about  $0.74 \text{ kg/L}$ . What is the volume of 10 grams of gasoline?
33. Use scientific notation to express
  - (a) 12 milligram in gram
  - (b) 42.2 kilometers in meters (distance of a marathon)
  - (c) 29 tonnes in gram
  - (d)  $0.062 \text{ mL}$  in liters
  - (e)  $20 \mu\text{m}$  in meters
34. Find convenient units for the quantities below. Example:  $4.8 \cdot 10^{-7} \text{ m} = 0.48 \mu\text{m} = 480 \text{ nm}$ .

- (a)  $3.2 \cdot 10^8 g$
- (b)  $0.5 \cdot 10^{-4} L$
- (c)  $8.8 \cdot 10^8 \mu m$
- (d)  $2.5 \cdot 10^7 m^2$
- (e)  $4 \cdot 10^{-10} m^3$

### Answers

1. (a)  $2,000 \text{ mm} = 200 \text{ cm} = 2 \text{ m} = 0.002 \text{ km}$   
 (b)  $0.0212 \text{ m}$   
 (c)  $21 \mu m$
2.  $375 \text{ mm} = 0.375 \text{ m}$
3. (a)  $5.3 \cdot 10^{-11} \text{ m}$   
 (b)  $6 \cdot 10^{-7} \text{ m}$   
 (c)  $2 \cdot 10^{-6} \text{ m}$   
 (d)  $10^{-4} \text{ m}$   
 (e)  $7.6 \cdot 10^{-4} \text{ m}$   
 (f)  $3 \cdot 10^{-2} \text{ m}$   
 (g)  $1.75 \cdot 10^0 \text{ m}$   
 (h)  $4.22 \cdot 10^4 \text{ m}$   
 (i)  $6.371 \cdot 10^6 \text{ m}$   
 (j)  $3.84 \cdot 10^8 \text{ m}$   
 (k)  $1.5 \cdot 10^{11} \text{ m}$
4.  $1.6 \text{ km}$
5.  $9,000.227,455 \text{ m}$
6.  $11 \text{ mm/hr}$
7.  $\approx 1.1 \text{ m}$
8.  $40 \text{ sq km}$
9.  $160 \text{ L}$
10.  $6.25 \text{ mL}$
11.  $2,500 \text{ mm}^2$
12.  $30,000 \text{ mm}^3$
13.  $1000$
14.  $2,500 \text{ cm}^2 = 0.25 \text{ m}^2$
15.  $600$
16.  $2,000,000 \text{ L}$
17. About  $21 \text{ cm}^2$
18.  $50 \text{ L}$
19. True
20.  $45,000 \text{ g}$
21.  $4 \text{ g}$
22.  $6,800 \text{ kg} = 6,800,000 \text{ g}$
23. (a)  $0.000,003,5 \text{ mg} = 0.003,5 \mu g$   
 (b) 286 million cells
24. area
25. power
26. (a) density  
 (b) velocity  
 (c) acceleration  
 (d) pressure
27. (a)  $3,237 \text{ J}$   
 (b)  $270 \text{ W}$
28.  $2,510,400 \text{ J} = 2.5104 \text{ MJ}$
29.  $19.3 \text{ g/cm}^3$
30. (a)  $3.5 \text{ g/cm}^3$   
 (b)  $0.4 \text{ mL}$
31.  $2 \text{ Pa}$
32.  $13.5 \text{ mL}$



33. (a)  $1.2 \cdot 10^{-3}\text{g}$   
 (b)  $4.22 \cdot 10^4\text{m}$   
 (c)  $2.9 \cdot 10^6\text{g}$   
 (d)  $6.2 \cdot 10^{-2}\text{L}$   
 (e)  $2 \cdot 10^{-5}$
34. (a) 320 tonnes  
 (b) 0.05 mL  
 (c)  $880\text{m} = 0.88 \text{ km}$   
 (d)  $25 \text{ km}^2$   
 (e)  $0.4 \text{ mm}^3$

### 3 Unit Conversion

Unit conversion is routine in biological work. For instance, it is reported that sailfish (*Istiophorus*) can swim 100 m in 4.8 seconds. How fast is this measured in miles per hour? Or, in a field study you determine that the average weight of tree squirrels in a state park is 1 lb and 5 oz. What is the equivalent value in gram? Or, you may have to figure out how many 4 mg doses of a vaccine can be obtained from a 1.5 gal supply.

This section will cover unit conversion methods. Before we get to the techniques, we will summarize the important relationships in tables. Conversion within the metric system was touched upon in the previous section and mastery of unit prefixes goes a long way for such problems.

#### 3.1 Customary U.S. Units

Historically, the common units used in the U.S. today were derived from British (imperial) units. They are closely related, but not quite identical.

1 ft	=	12 in
1 yd	=	3 ft
1 mi	=	5,280 ft
1 mi	=	1,760 yd
1 acre	=	43,560 ft <sup>2</sup>
1 mi <sup>2</sup>	=	640 acres
1 gal	=	4 qts
1 qt	=	32 oz
1 ft <sup>3</sup>	=	7.480,519 gal
1 barrel	=	42 gal
1 lb	=	16 oz
1 ton	=	2,000 lb

Figure 3: U.S. Units

Notice that the term *ounce* is used as a volume unit and as a mass unit. We speak of

*fluid ounces* when talking about volumes, and of *dry ounces* for masses.

Other customary units in the U.S. include *horsepower* (hp) and *British thermal units* (BTU). One horsepower (hp) is the power required to raise 550 lb at the rate of 1 ft/sec against gravity, and one BTU is the energy needed to raise the temperature of 1 lb of water by one degree Fahrenheit.

### 3.2 U.S. and Metric Units

The data are summarized in two tables.

Things that are commonly expressed in feet or yards should be converted to meters, miles should be converted to kilometers, and inches to centimeters. Areas of the size of acres should be expressed in hectares, a liter is about a quart, a pound is about one half of a kilogram, and imperial and metric tonnes are about the same.

1 m	=	3.280,840 ft
1 m	=	1.093,613 yd
1 km	=	0.621,371 mi
1 cm	=	0.393,701 in
1 m <sup>2</sup>	=	10.764 ft <sup>2</sup>
1 km <sup>2</sup>	=	0.386,102 mi <sup>2</sup>
1 ha	=	2.471,054 acres
1 L	=	1.056,688 qt
1 L	=	0.264,172 gal
1 kg	=	2.204,623 lb
1 g	=	0.035,274 oz
1 tonne	=	1.102,311 tons

Figure 4: U.S. to Metric

For **temperature** conversions we have

$$\begin{aligned}\text{°C} &= \frac{5}{9} (\text{°F} - 32^\circ) \\ \text{°F} &= 1.8 \text{°C} + 32^\circ \\ K &= \text{°C} + 273.15\end{aligned}$$

1 ft	=	0.3048 m
1 yd	=	0.9144 m
1 mi	=	1.609,344 km
1 in	=	2.54 cm
1 ft <sup>2</sup>	=	0.092,903 m <sup>2</sup>
1 mi <sup>2</sup>	=	2.589,988 km <sup>2</sup>
1 acre	=	0.404,686 ha
1 qt	=	0.946,353 L
1 gal	=	3.785,412 L
1 lb	=	0.453,592 kg
1 oz	=	28.349,523 g
1 ton	=	0.907,185 tonnes

Figure 5: Metric to U.S.

Finally, we note that

$$\begin{aligned}1 \text{ BTU} &= 1,055.056 \text{ J} \\ 1 \text{ cal} &= 4.184 \text{ J} \\ 1 \text{ hp} &= 745.7 \text{ W}\end{aligned}$$

BTU stands for British thermal unit and hp is horsepower.

### 3.3 Conversion Methods

There are basically three methods that can be applied to convert between units. We illustrate all three for the problem of converting 72 km/h to m/sec. Further examples are given in the next section.

#### 3.3.1 Substitution

**Example:**

$$\frac{72 \text{ km}}{\text{hr}} = \frac{72 \times 1,000 \text{ m}}{3,600 \text{ sec}} = 20 \text{ m/sec}$$

As the name indicates, 1 km was substituted by 1,000 m, and one hour was replaced by 3,600 seconds.

### 3.3.2 Conversion Ratios

This is probably the most popular conversion method.

**Example:**

$$\begin{aligned} & \frac{72 \text{ km}}{\text{hr}} \\ = & \frac{72 \cancel{\text{km}}}{\cancel{\text{hr}}} \times \frac{1,000 \text{ m}}{\cancel{\text{km}}} \times \frac{1 \cancel{\text{hr}}}{60 \cancel{\text{min}}} \times \frac{1 \cancel{\text{min}}}{60 \text{ sec}} \\ = & \frac{72 \times 1,000 \text{ m}}{60 \times 60 \text{ sec}} = 20 \text{ m/sec} \end{aligned}$$

The same calculation is frequently expressed in box form:

72 km	1,000 m	1 hr	1 min
hr	km	60 min	60 sec

The idea behind this method is multiplication by 1. For instance,

$$1 \text{ km} = 1,000 \text{ m}$$

and therefore

$$\frac{1,000 \text{ m}}{1 \text{ km}} = 1$$

The same goes for

$$\frac{1 \text{ hour}}{60 \text{ min}} = 1 \quad \text{or} \quad \frac{1 \text{ min}}{60 \text{ sec}} = 1$$

### 3.3.3 Conversion Factors

All values found in the conversion tables are conversion factors.

In our example we have to calculate factor for conversion from km/h to m/sec ourselves (substitution).

$$\frac{\text{km}}{\text{hr}} = \frac{1,000 \text{ m}}{3,600 \text{ sec}} = \frac{5 \text{ m}}{18 \text{ sec}} = 0.278 \text{ m/sec}$$

Once the factor is established, we can calculate

$$72 \text{ km/h} = 72 \times 0.278 \text{ m/sec} = 20 \text{ m/sec}$$

### 3.4 Worked Problems

1. **Sports:** How many acres are in an NFL football field (playing area plus end zones)?

Solution (Conversion Ratios): A football field is 100 yards long and 160 feet wide, and the end zones are 10 yards each.

$$\begin{aligned} & (100 + 2 \times 10) \text{ yards} \times 160 \text{ feet} \\ = & 19,200 \cdot \text{yd} \cdot \text{ft} \cdot \frac{3 \text{ ft}}{\text{yd}} \cdot \frac{\text{acres}}{43,560 \text{ ft}^2} \\ = & 1.32 \text{ acres} \end{aligned}$$

2. **Highway:** How many miles are the equivalent of 500 feet?

Solution (Conversion Ratios):

$$\begin{aligned} 500 \text{ ft} &= 500 \text{ ft} \times \frac{\text{mi}}{5,280 \text{ ft}} \\ &= \frac{500}{5,280} \text{ mi} = 0.0947 \text{ mi} \\ &\approx 0.1 \text{ mi} \end{aligned}$$

3. **Land Management:** Convert square miles to hectares. What is the conversion factor?

Solution:

$$\begin{aligned} (1 \text{ mi})^2 &= (1,609 \text{ m})^2 = 1.609^2 \text{ m}^2 \\ &= 2,588,881 \text{ m}^2 \cdot \frac{\text{ha}}{10,000 \text{ m}^2} \\ &\approx 259 \text{ ha} \end{aligned}$$

Thus, to convert from square miles to hectares, you should multiply by 259.

4. **Volume:** Convert 0.3 cubic inches to milliliters.

Solution:

$$\begin{aligned} 0.3 \text{ in}^3 &= 0.3 \times (2.54 \text{ cm})^3 \\ &= 0.3 \times 2.54^3 \text{ cm}^3 \\ &= 4.91 \text{ mL} \end{aligned}$$

Recall that cubic centimeters and milliliters are identical.

5. **Pediatrics:** A baby is born at 7 lb. and 9 oz. Express the birth weight in gram.

Solution:

$$\begin{aligned} 7 \text{ lb} + 9 \text{ oz} &= (7 \times 16 + 9) \text{ oz} \\ &= 121 \text{ oz} \times \frac{28.35 \text{ g}}{\text{oz}} = 3,430 \text{ g} \end{aligned}$$

## 6. Temperature Conversion

- (a) **Pediatrics:** A child is running a temperature of 102.5 degrees Fahrenheit. What is the equivalent Celsius value?
- (b) **Culinary:** A recipe for a loaf of bread calls for 220 degrees Celsius baking temperature. What is the American stove setting?

Solutions:

- (a)  $C = \frac{5}{9} (102.5 - 32) = 39.2$
- (b)  $F = 1.8 \cdot 220 + 32 = 428$

7. **Engineering:** How much power is required to raise two tons by five feet in two seconds?

Solution: The force is

$$\begin{aligned} F &= 2 \text{ tons} \cdot 9.81 \text{ m/sec}^2 \\ &= 2 \cdot 1,102.3 \cdot 9.81 \frac{\text{kg} \cdot \text{m}}{\text{sec}^2} \\ &= 21,627 \text{ N} \end{aligned}$$

The distance is 5 ft = 1.524 m. Hence, the power is given by

$$\begin{aligned} P &= \frac{21,627 \text{ N} \cdot 1.524 \text{ m}}{2 \text{ sec}} \\ &= 16,480 \text{ W} \\ &= 16,480 \text{ W} \cdot \frac{1 \text{ hp}}{745.7 \text{ W}} \\ &= 22.1 \text{ hp} \end{aligned}$$

The answer is 16,480 Watts, which is the equivalent of 22.1 horsepower.

8. **Medical:** A person's blood sugar is measured at 6 mmol/L (millimol per liter). Convert this value to mg/dL (milligram per deciliter).

Solution: mg/dL is the customary unit for blood sugar in the medical field. A deciliter is one tenth of a liter.

We need a little chemistry background to solve this problem. Glucose  $\text{C}_6\text{H}_{12}\text{O}_6$  has atomic mass 180 Da<sup>3</sup>. Therefore one mol of glucose has mass 180 g (=180,000 mg), and substitution results in

$$\begin{aligned} \frac{6 \text{ mmol}}{\text{L}} &= \frac{0.006 \text{ mol}}{10 \text{ dL}} \cdot \frac{180,000 \text{ mg}}{\text{mol}} \\ &= \frac{1080 \text{ mg}}{10 \text{ dL}} = 108 \text{ mg/dL} \end{aligned}$$

## 3.5 Exercises

Use scientific notation, if necessary.

- How many fluid ounces are in a cubic foot?
- How many miles are in 1500 feet?
- Convert 4 million inches to miles.
- Your lab animals need 3 ounces of a liquid food supplement each day. How long will a five gallon supply last?
- How many dry ounces make up 0.7 tons?
- Convert 0.3 acre feet to gallons. Acre foot is a volume unit commonly used for large amounts of water.

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<sup>3</sup>The respective atomic mass values are 12 Da for carbon, 1 Da for hydrogen and 16 Da for oxygen. Thus, glucose has atomic mass  $6 \times 12 \text{ Da} + 12 \times 1 \text{ Da} + 6 \times 16 \text{ Da} = 180 \text{ Da}$ .

7. Federal law has reserved 100 acres of land to be left unharvested. If the trees grow at a density of 3 trees/1000 sq feet, how many trees have been protected from logging?
8. Doctors suggest to drink 8 cups of water per day. How many milliliters is this?
9. A 12 inch ruler is how many centimeters long?
10. Buffalo Mountain in Floyd County is 3,971 feet above sea level. What is its elevation expressed in meters?
11. How many milliliters are in a gallon of milk?
12. How many acres are in a 1 kilometer by 3 kilometer enclosure?
13. The average density of the Earth is  $5.52 \text{ g/cm}^3$ . What is this density in (a)  $\text{kg/m}^3$ , (b)  $\text{lb/ft}^3$ ?
14. Bob has a maximum speed of 14 mph, Eric has a maximum speed of 8 m/sec, John's maximum speed is 100 ft/min, and Ted has a maximum speed of 800 km/day. Who is the fastest?
15. The current land speed record was set by Andy Green with a Thrust SSS, and he reached 1224 kmh, and thus he became the first driver of a land vehicle to break the sound barrier. Suppose you could sustain this speed, and the roads were cleared off, how long would it take to drive the 2,790 miles from L.A. to New York City?
16. Find the largest value of the following masses: 0.1 pounds, 2 ounces, 30 grams, and 0.02 kilograms.
17. A particular chemical has been shown to have beneficiary effects on human cognitive ability at dosages of  $60\mu\text{g}$  per kg of body weight. However, this very same chemical is harmful at dosages over  $100\mu\text{g}$  per kg of body weight. What is a healthy dosage for a patient weighing 120 lbs? What dosage is harmful?
18. (a) Convert  $32^\circ\text{F}$ ,  $70^\circ\text{F}$ ,  $0^\circ\text{F}$  and  $1000^\circ\text{F}$  to Celsius.  
(b) Convert  $25^\circ\text{C}$ ,  $37^\circ\text{C}$ ,  $-12^\circ\text{C}$  and  $300^\circ\text{C}$  to Fahrenheit.  
(c) At what temperature do you get the same value on the Fahrenheit and on the Celsius scale?
19. (a) Find the conversion factor from mmol/L to mg/dL for glucose. I. e. convert 1 mmol/L to mg/dL.  
(b) Use the result from part (a) to express  
i.  $7.5 \text{ mmol/L}$   
ii.  $4 \text{ mmol/L}$   
in mg/dL.  
(c) What A patient's glucose level is 153 mg/dL. What is the equivalent in mmol/L?

### Answers

1. 957.5 oz.
2. 0.284 miles
3. 63.1 miles
4. 213 days
5. 22,400 oz.
6. 97,755 gal.
7. 13,068 trees

8. 1,893 mL
9. 30.48 cm
10. 1,210 m
11. 3,785.4 mL
12. 741.3 acres
13. (a) 5,520 kg/m<sup>3</sup> (b) 345 lbs/ft<sup>3</sup>
14. Ted
15. 3 hr 40 min 6 sec
16. 2 oz
17. 3.266 mg and 5.443 mg
18. (a) 0°C, 21.6°C, -17.8°C and 537.80°C.  
(b) 77°F, 98.6°F, 10.4°F and 572°F  
(c) -40°C = -40°F
19. (a) 1 mmol/L = 18 mg/dL  
(b) (i) 135 mg/dL (ii) 27 mg/dL  
(c) 8.5 mmol/L

## 4 A Little Geometry

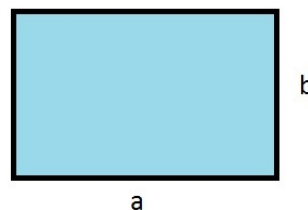
The surface area to volume ratio is an important concept in biology.

Leafs have a large surface area in relation to their volume, the surface area of apples is small compared to their volumes. A large surface area means a lot of interaction to the surrounding environment, which is beneficial for photosynthesis (sun light), but it also means a lot of exposure to the elements such as heat or extreme cold.

On a cellular level all nutrients must enter though the plasma membrane, and waste leaves that way. The size of the surface area limits the size (volume) of the cell: A cell cannot survive if it doesn't receive enough nourishment.

In this section we review some basics of geometry.

### 4.1 Two Dimensions: Area and Perimeter



The simplest two-dimensional object is a **rectangle**. If the sides are denoted by  $a$  and  $b$ , respectively, the area  $A$  becomes

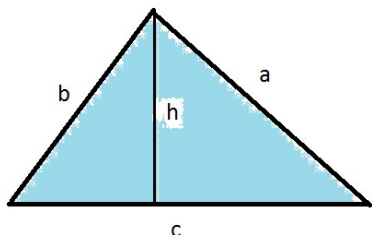
$$A = ab$$

and the perimeter  $p$  is

$$p = 2a + 2b$$

For the special case of a square ( $a = b = x$ ) we find that

$$A = x^2 \quad \text{and} \quad p = 4x$$



The area  $A$  of a **triangle** is one half of the product of base and height:

$$A = \frac{1}{2} ch$$

and the perimeter  $p$  is

$$p = a + b + c$$

For right triangles, and for those triangles only, we have the famous Pythagorean formula

$$c^2 = a^2 + b^2$$

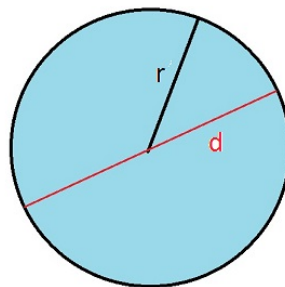
where  $c$  is the hypotenuse (the leg opposite the right angle). For such triangles one can use the remaining sides as base and height respectively, and the area becomes

$$A = \frac{ab}{2}$$

**The number**  $\pi$  is ubiquitous when dealing with circles and spheres. It is defined as the ratio between circumference and diameter of a circle

$$\pi = \frac{\text{circumference}}{\text{diameter}} = 3.141,592,653, \dots$$

which remains the same, regardless of the size of the circle.



The area  $A$  of a circle with radius  $r$  is

$$A = \pi r^2$$

and its circumference  $c$  is given by

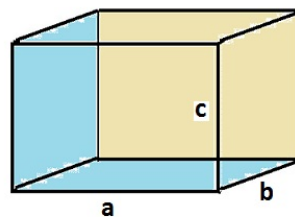
$$c = 2\pi r$$

If the diameter  $d$  is used, we find that

$$\begin{aligned} d &= 2r \\ c &= \pi d \\ A &= \frac{\pi d^2}{4} \end{aligned}$$

The units for areas must be the square of a length, such as square foot, square centimeters, et cetera.

## 4.2 Three Dimensions: Volume and Surface Area



The three-dimensional counterpart of a rectangle is a **box**. Its volume  $V$  is given by the product

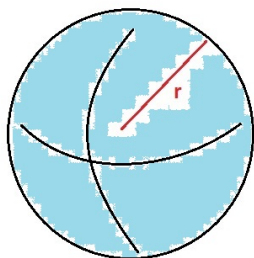
$$V = abc$$

and the surface area consists of six rectangular faces with total area

$$S = 2(ab + bc + ca)$$

For the special case of a cube ( $a = b = c = x$ ) we have

$$V = x^3 \quad \text{and} \quad S = 6x^2$$

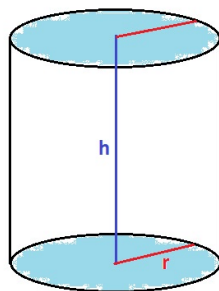


A **sphere** with radius  $r$  has volume

$$V = \frac{4}{3} \pi r^3$$

and surface area

$$S = 4\pi r^2$$



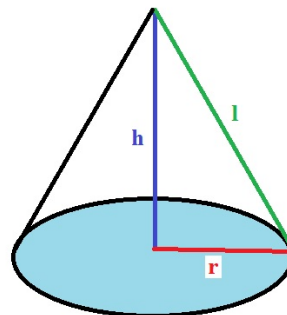
A **circular cylinder** is shaped like a can. It is determined by the radius  $r$  of the circles on top and bottom, and by its height  $h$ . The volume  $v$  of the cylinder is given by

$$V = \pi r^2 h$$

and its surface area  $S$  is

$$S = 2\pi r^2 + 2\pi r h = 2\pi r(r + h)$$

The term  $2\pi r^2$  accounts for the areas of the bottom piece and the lid, and the term  $2\pi r h$  measures the lateral surface area.



Finally we look at a **cone**. It has a circular base with radius  $r$ , and a height  $h$ . The lateral length  $l$  from the tip to the base perimeter is connected to the other quantities (right triangle, Pythagoras)

$$l^2 = r^2 + h^2$$

The volume of such a cone is given by

$$V = \frac{1}{3} \pi r^2 h$$

and its surface area is

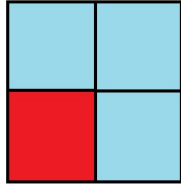
$$S = \pi r^2 + \pi r l$$

Volumes can be measured in cubic feet, cubic meters, and more importantly liters. Look for the third powers of length units.

### 4.3 Doubling Lengths and Surface Area to Volume Ratios

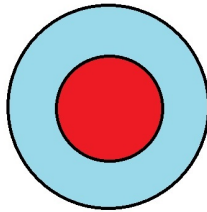
It is easy to see that the area of a square grows by a factor of 4 if the sides are doubled.





This can also be formally verified. We use  $x$  for the original length of the sides and  $x' = 2x$  for the new length. Then the respective areas are

$$\begin{aligned} A &= x^2 \\ A' &= (x')^2 = (2x)^2 = 4x^2 = 4A \end{aligned}$$

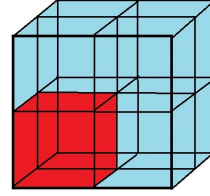


For circles it is not that obvious to tell by what factor the area is increased when the radius (or the diameter) is doubled. A formal calculation with  $r$  and  $r' = 2r$  as radii shows that

$$\begin{aligned} A' &= \pi(r')^2 = \pi(2r)^2 \\ &= \pi 4r^2 = 4A \end{aligned}$$

As a rule of thumb, we see that the area quadruples when the sides or the radius of an object are doubled. This is a characteristic of the two dimensional space (note that  $4 = 2^2$ ).

Things change in three dimensions. When we double the sides of a cube, the volume will become an eightfold of its original size. For instance, if you have a set of dice, all the same size, and you want to build a die one whose sides are twice as long, you need eight dice for your construction.



In an abstract calculation we find that

$$\begin{aligned} V' &= (x')^3 = (2x)^3 \\ &= 8x^3 = 8V \end{aligned}$$

and a similar computation can be used to see that the volume of a sphere also raises by an eightfold when the radius is doubled. This is a feature of three-dimensional objects ( $8 = 2^3$ ).

The **surface area to volume ratio** of a three-dimensional object is defined as

$$SVR = \frac{\text{surface area}}{\text{volume}} = \frac{S}{V}$$

For a cube whose sides have length  $x$ , we have  $V = x^3$  and  $S = 6x^2$ . Hence

$$SVR = \frac{S}{V} = \frac{6x^2}{x^3} = \frac{6}{x}$$

and we can calculate the surface area to volume ratio directly.

When the sides of the cube are doubled, the new surface area is  $S' = 4S$  and the new volume becomes  $V' = 8V$ , as we saw above. Therefore, the new surface area to volume ratio is

$$\begin{aligned} SVR' &= \frac{S'}{V'} = \frac{4S}{8V} \\ &= \frac{1}{2} \frac{S}{V} = \frac{1}{2} SVR \end{aligned}$$

and which is one half of the original ratio.

#### 4.4 Angular Speed and Centrifuges

The angular speed measures how fast something is revolving. This could be a search light

at an airport, the crankshaft in a car, an old vinyl record or a centrifuge, to name a few examples.

One full rotation is  $360^\circ$ , which in radians equals  $2\pi = 6.283\dots$ . We shall denote angular speed by  $\omega$ , and we will express it in radians per time.

**Example:** A search light makes one full revolution every 6 seconds. The angular speed becomes

$$\omega = \frac{2\pi}{5 \text{ sec}} = 1.257/\text{sec}$$

**Example:** The engine in a car revolves at 3,500 rpm (revolutions per minute). The angular speed becomes

$$\omega = \frac{3,500 \cdot 2\pi}{60 \text{ sec}} = 366.5/\text{sec}$$

In centrifuges we generate an acceleration (which becomes a force when applied to a mass). The parameters of importance are the radius of the rotor  $r$ , the angular speed  $\omega$  and the relative centrifugal force  $RCF$ , which relates the acceleration of the centrifuge to that of gravity. It is computed as

$$RCF = \frac{\omega^2 r}{g}$$

where  $g = 9.8 \text{ m/sec}^2$  is the constant of gravity.

**Example:** A bucket of water is attached to a rope so that the combined length of arm and rope is  $r = 1.5 \text{ m}$ . You spin the bucket such that one revolution takes 1.5 seconds. Here the angular speed is

$$\omega = \frac{2\pi}{1.5 \text{ sec}} = \frac{4\pi}{3} / \text{sec}$$

and the relative centrifugal force becomes

$$\begin{aligned} RCF &= \frac{\omega^2 r}{g} = \frac{\frac{16\pi^2}{9 \text{ sec}^2} \cdot 1.5 \text{ m}}{\frac{9.8 \text{ m}}{\text{sec}^2}} \\ &= \frac{16 \cdot \pi^2 \cdot 1.5}{9 \cdot 9.8} = 2.69 \end{aligned}$$

This is more than double the gravitational force, which is enough to keep the water from leaving the bucket in an up-side down position.

Centrifuges are used to separate cell material, and in order to isolate a certain substance, you have to adjust the rpm setting to obtain the correct RCF. Suppose that the rotor has radius  $r \text{ cm}$ , and that  $rpm$  is the rpm setting. Then

$$\omega = \frac{rpm \cdot 2\pi}{60 \text{ sec}} = \frac{rpm \cdot \pi}{30 \text{ sec}}$$

and

$$\begin{aligned} RCF &= \frac{\frac{rpm^2 \cdot \pi^2}{900 \text{ sec}^2} \cdot r \text{ cm}}{\frac{9.8 \text{ m}}{\text{sec}^2}} \\ &= \frac{\pi^2 rpm^2 r}{900 \cdot 9.8 \cdot 100} \\ &= 1.119 \cdot 10^{-5} rpm^2 r \end{aligned}$$

Therefore

$$rpm^2 = 89365.3 \frac{RCF}{r} \approx 90,000 \frac{RCF}{r}$$

and

$$rpm \approx 300 \sqrt{\frac{RCF}{r}}$$

**Example:** The radius of the JA-20 rotor in a Beckman J2-21 centrifuge is 10.8 cm. Isolation of chloroplasts requires a RCF of 1,300. The proper rpm selection is

$$rpm = 300 \sqrt{\frac{1,300}{10.8}} = 3,291 \approx 3,300$$

## 4.5 Worked Problems

1. The diameter of a **quarter**, as in a 25 cent coin, is 24.26 mm, its thickness is 1.75 mm, and its mass is 5.670 g.

(a) What is the area of each of the faces?

- (b) What is the volume of a quarter?  
 (c) What is its density?

Solutions:

- (a) We are looking for the area of a circle. The diameter is 24.26 mm; thus the radius is 12.13 mm, and the area becomes

$$\begin{aligned} A &= \pi(12.13 \text{ mm})^2 \\ &= 462.2 \text{ mm}^2 = 4.622 \text{ cm}^2 \end{aligned}$$

- (b) This is the volume of a cylinder. We have just computed the area of its base, and multiplication by the height yields the desired result (1.75 mm = 0.175 cm)

$$\begin{aligned} V &= 4.622 \text{ cm}^2 \times 0.175 \text{ cm} \\ &= 0.809 \text{ cm}^3 = 0.809 \text{ mL} \end{aligned}$$

- (c) Density is mass per volume, and we obtain

$$\rho = \frac{5.670 \text{ g}}{0.809 \text{ mL}} = 7.01 \text{ g/mL}$$

2. The bacterium **Escherichia coli** (E coli) is shaped like a rod (bacillus) and it is about 3  $\mu\text{m}$  long and 1  $\mu\text{m}$  in diameter. Use a cylindrical model to answer the questions below.

- (a) What are the volume and the surface area?  
 (b) What is the surface area to volume ratio?  
 (c) What is the mass of a single cell if we assume that the density is  $\rho = 1.1 \text{ g/mL}$ ?

Solutions:

- (a) We have  $r = 0.5 \mu\text{m}$  and  $h = 3 \mu\text{m}$ . Thus

$$\begin{aligned} V &= \pi \cdot 0.5^2 \cdot 3 \mu\text{m}^3 \\ &= 2.356 \mu\text{m}^3 \end{aligned}$$

and

$$\begin{aligned} S &= 2\pi \cdot 0.5(0.5 + 3) \mu\text{m}^2 \\ &= 11.00 \mu\text{m}^2 \end{aligned}$$

- (b)

$$SVR = \frac{11.00 \mu\text{m}^2}{2.356 \mu\text{m}^3} = 4.67 / \mu\text{m}$$

- (c) Mass = density  $\times$  volume. Thus

$$\begin{aligned} m &= 1.1 \text{ g/mL} \times 2.356 \mu\text{m}^3 \\ &= 2.6 \text{ g} \frac{\mu\text{m}^3}{\text{mL}} \end{aligned}$$

The tricky part is to simplify the ratio of cubic microns to milliliters (cubic centimeters). Since

$$\begin{aligned} 1 \mu\text{m} &= 0.001 \text{ mm} \\ &= 0.000,1 \text{ cm} = 10^{-4} \text{ cm} \end{aligned}$$

it follows that

$$\begin{aligned} 1 \mu\text{m}^3 &= (10^{-4})^3 \text{ cm}^3 \\ &= 10^{-12} \text{ cm}^3 \end{aligned}$$

and that

$$\frac{\mu\text{m}^3}{\text{mL}} = 10^{-12}$$

Therefore the mass of the cell is

$$m = 2.6 \cdot 10^{-12} \text{ g} = 2.6 \text{ pg}$$

Here pg stands for picogram.

3. Repeat the exercise for a **eukaryotic cell**, assuming that the cell is shaped like a sphere with radius  $24\mu\text{m}$ .

- (a) What are the volume and the surface area?
- (b) What is the surface area to volume ratio?
- (c) What is the mass of a single cell if we assume that the density is  $\rho = 1.1 \text{ g/mL}$ ?

Solution:

- (a) Here  $r = 12\mu\text{m}$  and from the formulas for spheres we find

$$\begin{aligned} V &= \frac{4}{3} \pi (12\mu\text{m})^3 \\ &= 7,238 \mu\text{m}^3 \end{aligned}$$

and

$$\begin{aligned} S &= 4\pi (12\mu\text{m})^2 \\ &= 1,810\mu\text{m}^2 \end{aligned}$$

- (b)

$$SVR = \frac{1,810\mu\text{m}^2}{7,238 \mu\text{m}^3} = 0.25/\mu\text{m}$$

In comparison to the bacterium we see that the surface area to volume ratio is much smaller for our eukaryote. This requires a higher degree of sophistication on part of eukaryotic cell in order to move matter around inside the cell, and to pass it through the plasma membrane.

- (c) The mass of the cell becomes (multiply mass and density)

$$\begin{aligned} m &= 1.1 \text{ g/mL} \times 7,238 \mu\text{m}^3 \\ &= 7,962 \text{ g} \frac{\mu\text{m}^3}{\text{mL}} \\ &= 7,962 \times 10^{-12} \text{ g} \\ &= 7.962 \times 10^{-9} \text{ g} \approx 8 \text{ ng} \end{aligned}$$

Here we are dealing in nanograms.

4. Find a formula for the surface area to volume ratio for a sphere.

Solution: We apply the formulas for a sphere and obtain

$$SVR = \frac{4\pi r^2}{\frac{4}{3} \pi r^3} = \frac{3}{r}$$

Here are some useful applications of this result:

- (a) For the eukaryotic cell with  $r = 12\mu\text{m}$  we get  $SVR = \frac{3}{12\mu\text{m}} = 0.25/\mu\text{m}$ , which confirms the result of the previous exercise.
- (b) The formula shows that doubling the radius will reduce the surface area to volume ratio by 50%.
- (c) Suppose you have an estimate of the minimum surface to volume ratio, which a cell can handle. Then you can calculate the maximum radius of a spherical cell from

$$r = \frac{3}{SVR}$$

5. A **dump truck** can carry about 10 cubic yards of dirt.

- (a) Given that dirt has a density of  $1.6 \text{ g/cm}^3$ , what is the weight (mass) of one truck load?
- (b) Excavation for a building site requires the removal of  $(42\text{ft}) \times (29\text{ft}) \times (8\text{ft})$  of dirt. How many truck loads are required?

Solutions:

- (a) First we convert 10 cubic yards to cubic centimeters:

$$\begin{aligned} 10 \text{ yd}^3 &= 10 \times (91.44 \text{ cm})^3 \\ &= 7.646 \times 10^6 \text{ cm}^3 \end{aligned}$$

Hence, the mass becomes (volume times density)

$$\begin{aligned} m &= 7.646 \times 10^6 \text{ cm}^3 \times 1.6 \frac{\text{g}}{\text{cm}^3} \\ &= 12.2 \times 10^6 \text{ g} = 12.2 \times 10^3 \text{ kg} \\ &= 12.2 \text{ tonnes} = 13.5 \text{ tons} \end{aligned}$$

- (b) The volume of dirt on our site is

$$\begin{aligned} V &= 42 \times 29 \times 6 \text{ ft}^3 \\ &= 7,308 \text{ ft}^3 \times \frac{\text{yd}^3}{27 \text{ ft}^3} \\ &= 270.67 \text{ yd}^3 \end{aligned}$$

Thus, approximately 27 loads are required. It's up to the driver to decide what to do about the extra 0.67 cubic yards.

6. **Oil Spill.** A circular oil spill in the Gulf of Mexico covers 24 square miles.

- (a) What is the diameter of the polluted area?  
 (b) Suppose that on average the oil layer is an eighth of an inch thick, how many gallons have been spilled?

Solutions:

With wind and ocean currents, a real oil spill will never look like a circle; we just use this model as an approximate shape.

- (a) We know the area, and we want to determine radius and diameter.

$$24 \text{ mi}^2 = A = \pi r^2$$

Therefore

$$r^2 = \frac{24 \text{ mi}^2}{\pi} = 7.64 \text{ mi}^2$$

and

$$r = \sqrt{7.64 \text{ mi}^2} = 2.76 \text{ mi}$$

The diameter is twice the radius:  
 $d = 5.5$  miles.

- (b) Asking for gallons means that we are looking for a volume. We are dealing with a huge circle covered with a layer of oil. Thus the formula for the volume of a cylinder applies (this cylinder is very flat).

$$V = 24 \text{ mi}^2 \times \frac{1}{8} \text{ in}$$

We still have to convert the product of square miles and inches to gallons. We use feet as a common comparison unit for miles and inches, and then convert to gallons and barrels.

$$\begin{aligned} V &= \frac{24}{8} \text{ mi}^2 \times \text{in} \\ &= 3 \times (5,280 \text{ ft})^2 \times \frac{1}{12} \text{ ft} \\ &= 6.97 \times 10^6 \text{ ft}^3 \times \frac{7.48 \text{ gal}}{\text{ft}^3} \\ &= 52.1 \times 10^6 \text{ gal} \\ &= 1.24 \times 10^6 \text{ barrels} \\ &= 1.24 \text{ million barrels} \end{aligned}$$

One barrel equals 42 gallons.

7. **Water Resources.** A river is 1,800 feet wide and the average depth is 20 feet. If the water flows at 2.5 mph on average, then how many gallons flow down the river each second?

Solution:

The flux is the area of the cross section (rectangle) multiplied by the velocity.

$$\begin{aligned} F &= 1,800 \text{ ft} \times 20 \text{ ft} \times 2.5 \frac{\text{mi}}{\text{hour}} \\ &= 90,000 \text{ ft}^2 \frac{\text{mi}}{\text{hr}} \end{aligned}$$

Substituting 1 mi = 5,280 ft and 1 hr = 3,600 sec we obtain

$$\begin{aligned} F &= \frac{90,000 \times 5,280}{3,600} \frac{\text{ft}^3}{\text{sec}} \\ &= 132,000 \frac{\text{ft}^3}{\text{sec}} \times \frac{7.48 \text{ gal}}{\text{ft}^3} \\ &= 987,000 \times \frac{\text{gal}}{\text{sec}} \end{aligned}$$

Roughly, this river transports a million gallons of water each second!

8. An artificial **pond in a zoo** has a circular boundary with a 40 meter diameter. The depth decreases linearly from zero to 5 meters in the center. How much water does it contain? Assume that it is shaped like a cone (up-side down).

Solution:

The data for the cone are  $r = 20$  m and  $h = 5$  m. Thus, the volume becomes

$$\begin{aligned} V &= \frac{1}{3} \pi (20 \text{ m})^2 5 \text{ m} \\ &= 2,094 \text{ m}^3 \end{aligned}$$

This is the equivalent of about 2 million liters.

9. The *Rotor* is an **amusement park ride**. It consists of a large upright barrel spinning at fast rates. Once the barrel has attained full speed, the floor is retracted, leaving the riders stuck to the wall of the drum. At the end of the ride cycle, the

drum slows down and gravity takes over, and the riders slide down the wall slowly.

Suppose that you stand 3 meters from the center and that the barrel makes 30 revolutions per minute.

- How fast are you moving, and
- what relative centrifugal force (RCF) do you experience?

Solution:

- The circumference of the rotor is  $2\pi \cdot 3$  m and you travel this distance 30 times in a minute, which makes for a speed of

$$s = \frac{2\pi \cdot 3 \cdot 30 \text{ m}}{60 \text{ sec}} = 3\pi \text{ m/sec}$$

The speed is 9.425 m/sec, which converted to miles per hour becomes 21.1 mph.

- The angular speed is  $\omega = \frac{30 \cdot 2\pi}{60 \text{ sec}} = \frac{\pi}{\text{sec}}$ , and therefore

$$RCF = \frac{\frac{\pi^2}{\text{sec}^2} \cdot 3 \text{ m}}{\frac{9.8 \text{ m}}{\text{sec}^2}} = 3.018$$

Three times the acceleration of gravity will keep the riders glued to the wall!

## 4.6 Exercises

- A circular plot at a garden show has circumference 75 ft. What is its area?
- By how much does the area of a square change, if all sides are increased by a factor of 5?
  - By how much does the volume of a cube change, if all sides are increased by a factor of 5?

3. A farmer puts up a fence with perimeter 2,500 ft to enclose a pasture. Assuming that the pasture is shaped like a square, how many acres is it?
4. Use  $1 \text{ L} = 1,000 \text{ cm}^3$ .
  - (a) A cube is to hold one liter, how long are the sides?
  - (b) A sphere is to hold one liter, what is its diameter?
5. Find a formula for the surface area to volume ratio of a circular cylinder.
6. An ant travels along the perimeter of a circle at a speed of 1.6 cm per second. What is the area of the circle if it takes the ant 3 minutes to complete one loop? Express the result in square meters.
7. A cylindrical test tube has diameter 1 cm. How tall is the column of liquid, measured from the bottom, if you fill the test tube with 6 mL of a liquid?
8. The distance from the equator to the pole is 10,000 km. What is the surface area of the Earth, assuming that it is a perfect sphere?
9. 40 million gallons of oil were spilled into the ocean during the Odyssey Oil Spill off the coast of Nova Scotia in 1988. Assuming that the layer of oil was 1/2 inch thick on average, what area did it cover?
10. A pond is shaped like a cone. Its diameter across is 56m and the depth in the center is 5m.
  - (a) What is the surface area of the the water?
  - (b) How many liters does the pond contain?
11. Suppose that a eukaryotic cell has the shape of a cube whose sides are  $24 \mu\text{m}$  each.
  - (a) What are the volume and the surface area?
  - (b) What is the surface to volume ratio?
  - (c) What is the mass of the cell, assuming that the cell density is  $\rho = 1.1 \text{ g/mL}$ ?
12. The cross sectional area of a small creek is  $2,500 \text{ cm}^2$ , and its average flow rate is 0.4 m/sec. How many liters of water pass through the creek each hour?
13. How long will it take for 0.1L of blood to pass through a vessel with diameter 3mm if the average flow rate is 12mm/sec?
14. How fast do second hand, the minute hand and the hour hand move on a clock? Express your answers in revolutions per minute and in radians per hour.
15. An object attached to a rope is spun with radius 10 m, and such that it takes four seconds for one revolution.
  - (a) What is the relative centrifugal force?
  - (b) What is the RCF if the object spins twice as fast?
  - (c) What is the RFC if the radius is increased to 20 m (and it still takes 4 seconds for one revolution)?
16. The radius of the rotor in your centrifuge is 12.3 cm. It takes 10,000 RCF to isolate mitochondria. What is the required rpm setting?

## Answers

1. 447.6 sq. ft.
2. (a) 25 times as large  
(b) 125 times more
3. 9 acres
4. (a) 10 cm (b) 6.2 cm
5.  $SVR = \frac{2(r+h)}{rh} = 2\left(\frac{1}{r} + \frac{1}{h}\right)$
6.  $0.66 \text{ m}^2$
7. 7.6 cm
8. 509.3 million  $\text{km}^2$
9. 4.603 square miles
10. (a)  $2,463 \text{ m}^2$  (b) 4.105 million liters
11. (a)  $13,824 \mu\text{m}^3$  and  $3,456 \mu\text{m}^2$   
(b)  $0.25/\mu\text{m}$   
(c) 15.2 ng
12. 360,000 L
13. 19 min 39 sec
14. Second hand: 1 rpm and 377 radians per hour.  
Minute hand:  $1/60$  rpm and 6.28 radians per hour.  
Hour hand:  $1/720$  rpm and 0.524 radians per hour.
15. (a) 2.515  
(b) 10.1  
(c) 5.03
16. 8,553 rpm

## 5 A Little Algebra

The computations in the first sections involved some routine algebra. In this section we will review some basic algebra. Most of it should be familiar material.

The binomial formula

$$(p+q)^2 = p^2 + 2pq + q^2$$

is related to the Hardy-Weinberg Law from genetics.

### 5.1 General Laws

The commutative and associative laws are the basic rules of arithmetic. They work for addition

$$\begin{aligned}a+b &= b+a \\(a+b)+c &= a+(b+c)\end{aligned}$$

and for multiplication

$$\begin{aligned}a \cdot b &= b \cdot a \\(a \cdot b) \cdot c &= a \cdot (b \cdot c)\end{aligned}$$

There is nothing difficult here ( $3+8=8+3$ ).



The associative laws allow us to group terms anyway we like it. Since  $(a+b)+c = a+(b+c)$ , we might just omit the parentheses and write  $a+b+c$ . The same goes for multiplication:  $(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c$

The distributive law connects addition and multiplication:

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

In all formulas the letters  $a$ ,  $b$ ,  $c$  are variables. You may replace them by any numbers



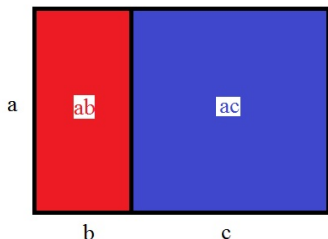


Figure 6:  $a \cdot (b + c) = a \cdot b + a \cdot c$

of your liking, and the equations will always work out.

The multiplication symbol  $\cdot$  is usually omitted. Everybody knows that  $3x$  means  $3 \cdot x$ .

The equality symbol ( $=$ ) means that two expressions have the same value. It should never be used as a mathematical punctuation symbol, separating different trains of thought! A hanging equality sign usually makes reference to a previous expression, as in

$$\begin{aligned} x^2 - 49 &= x^2 - 7x + 7x - 49 \\ &= x(x - 7) + 7(x - 7) \\ &= (x + 7)(x - 7) \end{aligned}$$

The chain of equalities is too long to be fitted into a single line, and we have to break it up.

We return to the **distributive law**. If you read it from left to right, it tells you how to expand an expression; when read from right to left, it is the factoring rule. The equality sign is a two-way street! For example, going left to right in

$$4y(x + 2) = 4xy + 8y$$

we distribute the  $4y$  with  $x$  and  $2$ , while going right to left we extract the common term  $y$  from both expressions.

As a very common special case of the distributive law we note the formula

$$a + ra = (1 + r)a$$

### Examples:

1. You add 18% tip to a \$ 45 restaurant bill. What is the total amount due?

$$\begin{aligned} 45 + 0.18 \cdot 45 &= (1 + 0.18) \cdot 45 \\ &= 1.18 \cdot 45 \end{aligned}$$

The point is not so much that the total bill is \$53.10, but that we can find this amount directly by multiplying the tab by 1.18 (rather than taking 18% first and then adding it to the total).

2. A \$300 recliner is reduced by 40%. How much do you have to pay?  
Rather taking 40% of 300 and then subtract it from 300, you can find the sales price by taking 60% of 300:

$$300 - 0.4 \cdot 300 = (1 - 0.4) \cdot 300 = 0.6 \cdot 300$$

The sales price is \$180, of course.

3. A population of 800 bugs increases by 25%. The new population is

$$800 + 0.25 \cdot 800 = 1.25 \cdot 800 = 1,000$$

The growth factor is  $1 + r = 1.25$

**Subtraction** is the inverse of addition. Specifically, if  $a$  and  $b$  are given numbers, then the number  $x$  with the property that

$$a = b + x$$

is defined to be  $x = a - b$ . Similarly, **division** is the inverse of multiplication. If  $a$  and  $b$  are given, then  $x = a \div b$  is the number such that

$$a = b \cdot x$$

This shows that division by zero is not permitted. Because if  $a \neq 0$  and  $b = 0$ , we get the contradiction  $a = 0 \cdot x = 0$ , no matter what value we take for  $x$ , and if both are zero, i.e. if  $a = 0$  and  $b = 0$ , then any number  $x$  will do the trick.

### 5.1.1 Fractions

A fraction is the result of a division:

$$\frac{a}{b} = a \div b$$

We have two important special cases

$$\frac{a}{1} = a \quad \text{and} \quad \frac{a}{a} = 1$$

Addition (or subtraction) of fractions demands that the denominators are identical, in which case

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Multiplication and division of fractions abide by the following rules

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} \\ \frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \end{aligned}$$

You multiply fractions by multiplication of the respective numerators and denominators; division of fractions follows a cross-multiplication.

Again, there are important special cases:

**Number times fraction**

$$a \cdot \frac{b}{c} = \frac{ab}{c}$$

**Cancelation of common factors**

$$\frac{ac}{bc} = \frac{a}{b}$$

You may also interpret this law as stating that multiplication of a fraction by the same number in numerator and denominator does not change its value.

**Three locations of a minus sign**

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

**Examples:**

1.  $\frac{8}{3} \cdot \frac{6}{4} = \frac{48}{12} = \frac{4 \cdot 12}{12} = 4$
2.  $\frac{4x}{3} \cdot \frac{x^2}{6} = \frac{4x \cdot 6}{3x^2} = \frac{8}{x}$
3.  $\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$
4.  $\frac{x - \frac{1}{x}}{x+1} = \frac{x^2 - 1}{x(x+1)} = \frac{(x-1)(x+1)}{x(x+1)}$   
 $= \frac{x-1}{x} = 1 - \frac{1}{x}$

### 5.1.2 Hierarchy of Operations

Hypothetically the quantity  $3 + 5 \cdot 6$  could be either

$$(3 + 5) \times 6 = 8 \times 6 = 48$$

or

$$3 + (5 \times 6) = 3 + 30 = 33$$

Which is correct, 48 or 33?

There is a convention about the order in which algebraic operations are carried out. In terms of priority we have the hierarchy

1. exponents
2. multiplication, division
3. addition, subtraction

Use parentheses if you want to break the order. Operations with the same priority are usually carried out left to right.

You may know this principle by the acronym PEMDAS (parentheses, exponents, multiplication, division, addition, subtraction).

In our example the answer 33 is correct. If we mean the calculation, which leads to 48, we have to write  $(3 + 5) \cdot 6$ .

We have used PEMDAS before without making a big fuss about it. For instance, in the distributive law the expression  $a \cdot b + a \cdot c$  means

$(a \cdot b) + (a \cdot c)$ , but knowing the order of operations, the parentheses are superfluous.

**Your calculator knows** the order of operations. If you enter

$$3 + 5 * 6 =$$

it will respond with 33.

But there are pitfalls! The calculator will compute what you type in, and not what you mean. Here are some **typical errors**:

1. 
$$\frac{20}{2 + 3}$$

If you type

$$20 \div 2 + 3 =$$

the calculator interprets this as

$$(20 \div 2) + 3 = 10 + 3 = 13$$

as division has priority over addition. The correct way is to use parentheses

$$20 \div (2 + 3) =$$

which will produce the right answer.

2. 
$$\frac{42}{3 \cdot 7}$$

Obviously,  $\frac{42}{3 \cdot 7} = \frac{42}{21} = 2$ . If you enter

$$42 \div 3 \cdot 7 =$$

the calculator will respond with 98, because (equal priority, left to right)

$$42 \div 3 \cdot 7 = 14 \cdot 7 = 98$$

The correct way to enter the expression is

$$42 \div (3 \cdot 7) =$$

## 5.2 FOIL, Binomial Formulas and the Pascal Triangle

Repeated application of the distributive law results in the famous foil formula:

$$\begin{aligned} (a + b)(c + d) &= a(c + d) + b(c + d) \\ &= ac + ad + bc + bd \\ &= F + O + I + L \end{aligned}$$

If  $a = c$  and  $b = d$ , the FOIL-result becomes

$$(a + b)^2 = a^2 + 2ab + b^2$$

and if  $b$  is replaced by  $-b$  it follows that

$$\begin{aligned} (a - b)^2 &= a^2 + 2a(-b) + (-b)^2 \\ &= a^2 - 2ab + b^2 \end{aligned}$$

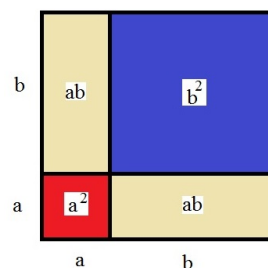


Figure 7:  $(a + b)^2 = a^2 + 2ab + b^2$

If in the FOIL formula  $a = c$  and  $d = -b$ , the middle terms will cancel and we find that

$$(a + b)(a - b) = a^2 - b^2$$

The last three results are known as *binomial formulas*. If you are comfortable with them, you can avoid lengthy FOIL-exercises:

$$\begin{aligned} (x + 5)^2 &= x^2 + 10x + 25 \\ (8 - x)^2 &= 64 - 16x + x^2 \\ 9 - x^2 &= 3^2 - x^2 = (3 - x)(3 + x) \end{aligned}$$

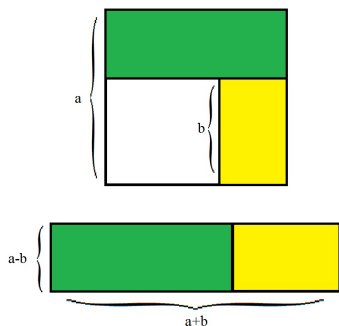


Figure 8:  $a^2 - b^2 = (a - b)(a + b)$

Higher order binomial formulas can be confirmed by repeated multiplication and FOIL-expansions. The third order binomial formulas are

$$\begin{aligned}(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\ a^3 - b^3 &= (a - b)(a^2 + ab + b^2)\end{aligned}$$

### 5.2.1 Pascal's Triangle

Pascal's triangle is a useful tool when working with binomial formulas.

If you look at the formula

$$(a + b)^2 = a^2 + 2ab + b^2$$

you should notice that the powers of  $a$  decrease as you move from left to right, while those of  $b$  increase. When you add the exponents, the sum always equals 2 (use  $b^0 = 1$  and  $a^0 = 1$ ), and the pattern of coefficients 1-2-1 describes the formula completely.

The same can be observed in

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Here the exponents add to 3, and the coefficients exhibit a 1-3-3-1 pattern. Hence, if we knew the pattern of the coefficients, we could

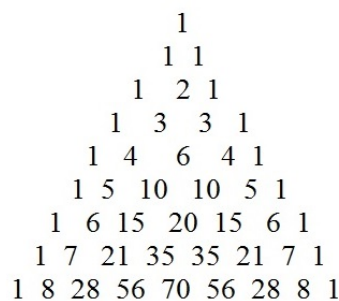


Figure 9: Pascal's Triangle

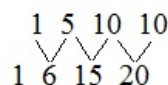
easily write a formula for  $(a+b)^n$  for any power  $n$ , and Pascal's triangle will do just that.

The Pascal triangle looks like a pyramid, with ones on the outside. The numbers on the inside are obtained by adding the numbers which are directly above the space in the previous line.

For instance, if you look at the figure and you take the line beginning with

$$1 \ 6 \ 15 \ 20 \ \dots$$

you see that 6 is obtained by adding 1 and 5, which are located above the 6, and then take  $15=5+10$ ,  $20=10+10$ , and so on. Just add the two numbers, which are directly above.



Inspection of the Pascal triangle reveals that the coefficients of  $(a+b)^2$  show up in the third line, those of  $(a+b)^3$  in the fourth. As it turns out, the coefficients of  $(a+b)^n$  are the values in the  $n$ -th line, provided that you start counting with zero. For instance

$$\begin{aligned}(a + b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 \\ &\quad + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

or

$$\begin{aligned}
 & (x - 3)^4 \\
 = & x^4 + 4x^3(-3) + 6x^2(-3)^2 \\
 & + 4x(-3)^3 + (-3)^4 \\
 = & x^4 - 12x^3 + 54x^2 - 108x + 81
 \end{aligned}$$

FOIL it, if you don't trust these results!

### 5.3 Hardy-Weinberg Equilibrium

We divert from the algebra review for a moment, and look at a biological application.

The Hardy-Weinberg Law has its origin in population genetics. Instead of studying genetics for just one individual organism, we keep track of all alleles for a particular gene in the population.

Suppose that a gene has alleles  $A$  and  $a$ . We denote by  $p$  the proportion of the allele  $A$  in the population, and by  $q$  the proportion of the allele  $a$ . Then, by design, we must have

$$p + q = 1$$

Let's say that for example the allele  $A$  has

	<b>A</b> 60%	<b>a</b> 40%
<b>A</b> 60%	<b>AA</b> 36%	<b>Aa</b> 24%
<b>a</b> 40%	<b>Aa</b> 24%	<b>aa</b> 16%

Figure 10: Punnett Square

frequency  $p = 0.6 = 60\%$  and that  $a$  has frequency  $q = 0.4 = 40\%$ . If mating was solely based on the laws of probability, then the chances

that a gamete has genotype  $AA$  is 36% (60% from the egg and 60% from the sperm). The genotype  $Aa$  can occur in two ways: Allele  $A$  comes from the egg (60% probability), and  $a$  comes from the sperm (40% probability), or vice versa. In each case the probability is  $0.24 = 0.6 \cdot 0.4$  and overall, the genotype  $Aa$  has frequency 42%. Finally, genotype  $aa$  has frequency  $16\% = 0.4^2$ .

This reasoning works for any values of  $p$  and  $q$ , and we can summarize the result as

Genotype	Frequency
$AA$	$p^2$
$Aa$	$2pq$
$aa$	$q^2$

Since  $p + q = 1$ , it follows from the binomial formula that

$$p^2 + 2pq + q^2 = (p + q)^2 = 1^2 = 1$$

This is the Hardy-Weinberg principle. If the relative frequencies of genotypes follow this equation, we say that the population is in Hardy-Weinberg equilibrium.

Recall, that the distribution of genotypes was based on probabilities alone. This presumes that there are no mutations, there is no natural selection and no gene flow, that mating is random and that the population is large enough to justify probabilistic reasoning. When the population is in Hardy-Weinberg equilibrium it is not evolving with respect to that gene.

**Example: Wildflowers.** A study of wildflowers on a meadow finds 320 red flowers (genotype  $RR$ ), 160 pink flowers ( $Rr$ ), and 20 white flowers ( $rr$ ).

Genotype	Phenotype	Freq.	Perc.
$RR$	red	320	64%
$Rr$	pink	160	32%
$rr$	white	20	4%
		500	100%

The data are summarized in the table. Here the alleles  $R$  and  $W$  determine the color. First we calculate the frequency of the alleles in the population. Red flowers carry two  $R$  alleles, pink flowers only have one  $R$  allele. Therefore

$$\begin{aligned} R : \quad & 320 \cdot 2 + 160 = 800 \\ r : \quad & 160 + 2 \cdot 20 = 200 \end{aligned}$$

The values for  $p$  and  $q$  become

$$\begin{aligned} p &= \frac{800}{1000} = 0.8 \quad \text{and} \\ q &= \frac{200}{1000} = 0.2 \end{aligned}$$

It now follows that

$$\begin{aligned} p^2 &= 0.8^2 = 0.64 = 64\% \\ 2pq &= 2 \cdot 0.8 \cdot 0.2 = 0.32 = 32\% \\ q^2 &= 0.2^2 = 0.04 = 4\% \end{aligned}$$

Since the predicted percentages match the observations, we conclude that the population is in Hardy Weinberg equilibrium.

**Example: Rock Mice.** In a study of rock mice researchers counted 374 brown mice ( $BB$ ), 344 tan mice ( $Bb$ ) and 32 white mice ( $bb$ ).

First we determine  $p$  and  $q$ . We have

$$\begin{aligned} B : \quad & 374 \cdot 2 + 344 = 1092 \\ b : \quad & 344 + 2 \cdot 32 = 408 \end{aligned}$$

With 750 mice we have a total of 1,500 alleles and the frequencies are

$$\begin{aligned} p &= \frac{1092}{1500} = 0.728 \quad \text{and} \\ q &= \frac{408}{1500} = 0.272 \end{aligned}$$

The equilibrium percentages are

$$\begin{aligned} p^2 &= 0.728^2 = 53.0\% \\ 2pq &= 39.6\% \\ q^2 &= 7.4\% \end{aligned}$$

We compare observation and predictions in a table.

		observed		H W equilib.	
BB	brown	373	49.9%	53.0%	397.5
Bb	tan	344	45.9%	39.6%	297.0
bb	white	32	4.2%	7.4%	55.5
		750	100%	100%	750

The side-by-side comparison shows that the observed tan phenotype is over-represented, compared to the Hardy Weinberg predictions, at the expense of the other phenotypes, and we have a heterozygous advantage.

Further analysis is required to determine whether the findings are statistically significant, or whether they can be explained by chance alone.

**Example: PKU.** Phenylketonuria (PKU) is a recessive disease. When left untreated it can lead to brain damage, or even death. In the United States one in 15,000 babies is born with PKU (for Caucasian babies the ratio is worse, it is 1:10,000).

We assume that the population is in Hardy Weinberg equilibrium in regard to PKU. If  $A$  is the dominant allele, and  $a$  is the recessive allele, we know that  $aa$  has frequency

$$q^2 = \frac{1}{15,000}$$

Thus,

$$q = \sqrt{\frac{1}{15,000}} = 0.008,165$$

and consequently,

$$p = 1 - q = 0.991,835$$

The relative frequency of  $Aa$  then becomes

$$2pq = 0.0162$$

and we conclude that about 1.62% of the population are heterozygotes, they carry the PKU

allele without showing symptoms of the disease.

## 5.4 Exponents

For any number  $a$  and any positive integer we have the familiar definition

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

For example,

$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$$

or

$$\begin{aligned} \left(-\frac{5}{4}\right)^3 &= \left(-\frac{5}{4}\right) \times \left(-\frac{5}{4}\right) \times \left(-\frac{5}{4}\right) \\ &= -\frac{5^3}{4^3} = -\frac{125}{64} = -1.953,125 \end{aligned}$$

The special case  $a^1 = a$  is obvious, but worth mentioning.

There are two fundamental rules for exponents, and we begin by looking at examples. First we have

$$5^3 \cdot 5^4 = (5 \cdot 5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5) = 5^8$$

and we see that in a product we add the exponents. Secondly

$$\begin{aligned} (5^3)^4 &= (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \\ &= 5^{12} \end{aligned}$$

tell us that when the exponents are stacked, we multiply them.

The associative law lets us disregard the parentheses, and there is nothing special about the numbers 3, 4 and 5. The underlying basic laws for exponents are

$$\begin{aligned} a^m a^n &= a^{m+n} \\ (a^m)^n &= a^{mn} \end{aligned}$$

Routine applications of these rules are calculations such as

$$\begin{aligned} 10^6 \cdot 10^9 &= 10^{15} & \text{or} \\ (x^2)^2 &= x^4 \end{aligned}$$

Our definition of  $a^n$  did require that  $n$  must be a positive integer. If we want to use  $n = 0$ , or allow for negative exponents, we need to expand our definition, and we should do it in such a way, that the basic laws remain valid.

We begin with  $a^0$  and set  $x = a^0$ . Then

$$ax = a \cdot a^0 = a^{1+0} = a$$

Division by  $a$  shows that  $x = 1$ , and we get  $a^0 = x = 1$ . This only works if  $a \neq 0$  and we say that  $0^0$  is undefined<sup>4</sup>.

Now we look at negative exponents. Let  $a \neq 0$  and set  $x = a^{-n}$ . Then

$$xa^n = a^{-n}a^n = a^{-n+n} = a^0 = 1$$

Therefore,

$$a^{-n} = x = \frac{1}{a^n}$$

The important special case  $n = -1$  results in

$$a^{-1} = \frac{1}{a^n} \quad \text{and} \quad \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

There is a host of other rules governing operations with exponents. But they can be easily derived from the basic concepts. For instance, We have

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n} = a^{m-n}$$

---

<sup>4</sup>If  $a = 0$ , the equation becomes

$$0 \cdot x = 0$$

and  $x$  could be any real number. For this reason we say that  $0^0$  is undefined. Many formulas and results in Calculus become easier to state if the convention  $0^0 = 1$  is used. See whether your calculator has an opinion on this.

as well as

$$\begin{aligned}(ab)^n &= (ab) \cdot (ab) \cdots (ab) \\ &= (a \cdot a \cdots a) \cdot (b \cdot b \cdots b) = a^n b^n\end{aligned}$$

and

$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b} \\ &= \frac{a \cdot a \cdots a}{b \cdot b \cdots b} = \frac{a^n}{b^n}\end{aligned}$$

**Examples:**

1.  $(4 \cdot 10^5)^2 = 4^2 \cdot (10^5)^2 = 16 \cdot 10^{10}$   
 $= 1.6 \cdot 10 \cdot 10^{10} = 1.6 \cdot 10^{11}$
2.  $\frac{3^{-4}}{3^8} = 3^{-4-8} = 3^{-12}$
3.  $\left(\frac{3}{5x^2}\right)^0 = 1$
4.  $\left(\frac{3}{x^2}\right)^4 = \frac{3^4}{(x^2)^4} = \frac{81}{x^8}$

### 5.4.1 Fractional Exponents

It is not uncommon to encounter expressions like  $12^{4/3}$ ,  $10^{1.2}$  or  $0.25^{-1/2}$ . Your calculator will give you numerical values for any of these. Why? How does the calculator evaluate these expressions?

Again, we need to come up with a definition which is consistent with the basic rules.

First we look at an example, and we set  $x = 12^{1/3}$ . Then

$$x^3 = \left(12^{1/3}\right)^3 = 12^{\frac{1}{3} \cdot 3} = 12^1 = 12$$

Thus,  $x^3 = 12$ , and therefore  $x$  must be the cubed root of 12:

$$12^{1/3} = x = \sqrt[3]{12}$$

If we were interested in  $12^{4/3}$ , we could compute

$$12^{4/3} = (12^{1/3})^4 = (\sqrt[3]{12})^4$$

There is nothing special about 12, 3 and 4, and as a general rule we find that

$$a^{1/n} = \sqrt[n]{a}$$

and that

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

If  $n = 2$ , we are taking a square root, and we know that in this case  $a$  cannot be negative. On the other hand, it is possible to take cubed roots of negative numbers as in

$$\sqrt[3]{-64} = -4$$

because  $(-4)^3 = -64$ . We don't want to get too technical here, and we adopt the philosophy that when in doubt we take  $a \geq 0$ .

It is worthwhile to point out that there is no  $\pm$  when you compute square roots. For instance,  $\sqrt{25} = 5$  and not  $\sqrt{25} = \pm 5$ . Undoubtedly, the equation

$$x^2 = 25$$

is solved by either  $x = 5$  or by  $x = -5$ , but this is a different story. When we talk about the square root, we mean the principal root, and this is the non-negative answer. Your calculator sees it this way too, there is no  $\pm$  popping up when you compute things like  $\sqrt{25}$ .

For the other introductory examples we find that

$$\begin{aligned}10^{1.2} &= 10^{6/5} = \left(\sqrt[5]{10}\right)^6 = 1.585^6 \\ &= 15.85\end{aligned}$$

and

$$\begin{aligned}0.25^{-1/2} &= (0.25^{-1})^{1/2} = 4^{1/2} = \sqrt{4} \\ &= 2\end{aligned}$$



Most **calculators** use caret key  $\wedge$  to take exponents. If your calculator does not have a (convenient) key to take the  $n$ -th order root of a number, you can always use the  $1/n$ -th power as an alternative, as in

$$\sqrt[5]{10} = 10^{1/5} = 10^{0.2} = 1.585$$

## 5.5 Worked Problems

1. A population of 16,000 bees declines by 12.5%. What is the new population?

Solution:

$$\begin{aligned} & 16,000 - 0.125 \cdot 16,000 \\ &= (1 - 0.125) \cdot 16,000 \\ &= 0.875 \cdot 16,000 \\ &= 14,000 \end{aligned}$$

The new population is 14,000 bees and the multiplier is 0.875.

2. Expand the expression  $-3(4 - x)$ .

Solution:

$$-3(4 - x) = -12 + 3x$$

Don't forget to distribute the negative sign.

3. Factor the expression  $x^2y^2 - 3xy^3$

Solution: Both terms have one factor of  $x$  and two factors of  $y$  in common:

$$\begin{aligned} & x^2y^2 - 3xy^3 \\ &= x \cdot x \cdot y \cdot y - 3 \cdot x \cdot y \cdot y \cdot y \cdot y \\ &= x \cdot y \cdot y(x - 3 \cdot y \cdot y) \\ &= xy(x - 3y) \end{aligned}$$

You don't need to show this much detail in your work! We only did this once to illustrate the details.

4. Expand and simplify the expression

$$(x - 2)(x + 2)(x^2 + 4)$$

Solution:

$$\begin{aligned} & (x - 2)(x + 2)(x^2 + 4) \\ &= [(x - 2)(x + 2)](x^2 + 4) \\ &= [x^2 + 2x - 2x - 4](x^2 + 4) \\ &= (x^2 - 4)(x^2 + 4) \\ &= x^4 + 4x^2 - 4x^2 - 16 \\ &= x^4 - 16 \end{aligned}$$

The solution becomes much shorter if you apply the formula  $(a - b)(a + b) = a^2 - b^2$  (twice).

5. Simplify the expression

$$\begin{aligned} \text{(a)} \quad & \frac{3}{8} - \frac{1}{6} \\ \text{(b)} \quad & \frac{25x^3 + 15x}{10x^2} \\ \text{(c)} \quad & \frac{x + \frac{1}{x}}{x} - \frac{1}{x^2}. \end{aligned}$$

Solutions

- (a) Here the least common denominator is 24, and we have

$$\begin{aligned} \frac{3}{8} - \frac{1}{6} &= \frac{3 \cdot 3}{8 \cdot 3} - \frac{1 \cdot 4}{6 \cdot 4} \\ &= \frac{9}{24} - \frac{4}{24} = \frac{9 - 4}{24} = \frac{5}{24} \end{aligned}$$

- (b) The numerator has the common factor  $5x$ , and we take advantage of it.

$$\begin{aligned} & \frac{25x^3 + 15x}{10x^2} \\ &= \frac{5x(5x^2 + 3)}{(5x)(2x)} = \frac{5x^2 + 3}{2x} \\ &= \frac{5x^2}{2x} + \frac{3}{2x} = \frac{5x}{2} + \frac{3}{2x} \end{aligned}$$

Both answers,  $\frac{5x^2+3}{2x}$  and  $\frac{5x}{2} + \frac{3}{2x}$  are acceptable solutions of the problem.

- (c) As a first step, we multiply numerator and denominator in the first term by  $x$ . This will get rid of the double fraction, and it also generates a common denominator for the two fractions. The rest is straightforward.

$$\begin{aligned} & \frac{x + \frac{1}{x}}{x} - \frac{1}{x^2} \\ = & \frac{\left(x + \frac{1}{x}\right)x}{x^2} - \frac{1}{x^2} \\ = & \frac{x^2 + 1}{x^2} - \frac{1}{x^2} = \frac{x^2 + 1 - 1}{x^2} \\ = & 1 \end{aligned}$$

The problem is even simpler once you realize that

$$\frac{x + \frac{1}{x}}{x} = 1 + \frac{1}{x^2}$$

6. Expand the expressions.

- (a)  $(2.4 + x)^2$   
 (b)  $(2x - 1)^3$   
 (c)  $(10 + x)^4$

Solutions: We apply the binomial formulas. Feel free to FOIL and check the results.

- (a)  $(2.4 + x)^2$   
 $= 2.4^2 + 2 \cdot 2.4 \cdot x + x^2$   
 $= 5.76 + 4.8x + x^2$   
 (b)  $(2x - 1)^3$   
 $= (2x)^3 - 3(2x)^2 + 3(2x) - 1$   
 $= 8x^3 - 12x^2 + 6x - 1$

- (c)  $(10 + x)^4$   
 $= 10^4 + 4 \cdot 10^3x + 6 \cdot 10^2x^2$   
 $+ 4 \cdot 10x^3 + x^4$   
 $= 10,000 + 4,000x + 600x^2$   
 $+ 10x^3 + x^4$

7. Simplify

- (a)  $(5 \cdot 10^{-1})^{-2}$   
 (b)  $\frac{x^5y^{-2}z}{x^3y^5z^3}$   
 (c)  $\frac{18x^3}{yz} \div \frac{9z^2}{x^2y}$

Solutions:

- (a)  $(5 \cdot 10^{-1})^{-2} = 5^{-2} \cdot (10^{-1})^{-2}$   
 $= \frac{1}{5^2} \cdot 10^2 = \frac{100}{25} = 4$   
 After all,  $5 \cdot 10^{-1} = \frac{5}{10} = \frac{1}{2}$  and  
 $(\frac{1}{2})^{-2} = 2^2 = 4.$   
 (b) Collect the powers for each variable separately:

$$\begin{aligned} \frac{x^5y^{-2}z}{x^3y^5z^3} &= x^{5-3}y^{-2-5}z^{1-3} \\ &= x^2y^{-7}z^{-2} = \frac{x^2}{y^7z^2} \end{aligned}$$

- (c) Cross-multiply and simplify:

$$\frac{18x^3}{yz} \div \frac{9z^2}{x^2y} = \frac{18x^3 \cdot x^2y}{yz \cdot 9z^2} = \frac{2x^5}{z^3}$$

8. Compute the values

- (a)  $8^{4/3}$   
 (b)  $9^{-3/2}$   
 (c)  $0.01^{-2.5}$   
 (d) 2560.25  
 (e)  $(\sqrt{125})^{4/3}$

You are encouraged to confirm all results on your calculator.

Solutions:

$$(a) 8^{4/3} = (\sqrt[3]{8})^4 = 2^4 = 16$$

$$(b) 9^{-3/2} = \frac{1}{9^{3/2}} = \frac{1}{(\sqrt{9})^3} = \frac{1}{3^3} = \frac{1}{27}$$

$$(c) 0.01^{-2.5} = (10^{-2})^{-2.5} = 10^{(-2) \cdot (-2.5)} = 10^5 = 100,000$$

(d) If we use that  $125 = 5^3$ , we obtain

$$\begin{aligned} (\sqrt{125})^{4/3} &= \left( (5^3)^{1/2} \right)^{4/3} \\ &= 5^{3 \cdot \frac{1}{2} \cdot \frac{4}{3}} = 5^2 = 25 \end{aligned}$$

9. Allele  $A$  is dominant over  $a$ . Plants of genotype  $A+$  (this means  $AA$  or  $Aa$ ) have round seeds, those with genotype  $aa$  have wrinkled seeds. If the relative frequency of  $A$  in the population is 70%, then what percentage of plants will have wrinkled seeds?

Solution: We know that  $p = 70\% = 0.7$ . Therefore  $q = 1 - p = 0.3$ , and  $q^2 = 0.09$ . Thus, the relative frequency of  $aa$  is  $q^2 = 9\%$ .

10. A population of 500 dogs is in Hardy-Weinberg equilibrium in regard to coat color. If 100 dogs are black ( $BB$ ), then how many dogs should be brown ( $Bb$ ), and how many are yellow ( $bb$ ).

Solution: We know that

$$p^2 = \frac{100}{500} = \frac{1}{5} = 0.2$$

Therefore

$$\begin{aligned} p &= \sqrt{0.2} = 0.447 \quad \text{and} \\ q &= 1 - 0.447 = 0.553 \end{aligned}$$

Since

$$\begin{aligned} 2 \cdot p \cdot q \cdot 500 &= 247.2 \quad \text{and} \\ q^2 \cdot 500 &= 152.8 \end{aligned}$$

we expect to have about 247 brown and about 153 yellow dogs.

## 5.6 Exercises

1. Expand the expressions

- (a)  $3(4 + x)$
- (b)  $-15(6 + x)$
- (c)  $-4(10 - x)$
- (d)  $(2 - x)(2 + x)$
- (e)  $3x(3 - x + y)$
- (f)  $(5 + 2x)^2$
- (g)  $(x - 3)^2$
- (h)  $x(x - 2)^2$

2. Factor the expressions

- (a)  $21 - 7x$
- (b)  $2xy - x^2$
- (c)  $144 - x^2$
- (d)  $4x^2 - 12x + 9$

3. Express as a single fraction and simplify

- (a)  $\frac{4}{3} - \frac{3}{4}$
- (b)  $x + \frac{1}{x}$
- (c)  $\frac{3}{x} \times x^2$
- (d)  $\frac{x+1}{x-1} - \frac{x-1}{x+1}$
- (e)  $\frac{x^2}{8} \times \frac{6}{x}$
- (f)  $\frac{x^2}{8} \div \frac{6}{x}$

4. SuperSale! All items are 45% off! What should you expect to pay for a \$139.95 camera?
5. The rat population in a city increases by 13% per year. Beginning with 2,300 rats, how many rats do you expect the following year?
6. The population of African lions is currently estimated at 32,000. What is their population next year, if it declines by 8% annually?
7. A circle of radius  $r = 25$  meters is changed by  $x$  meters (the new radius is  $r = 25 + x$  meters). What is the resulting area? Find a general expression involving  $x$ , and check your result when  $x = 2\text{m}$ ,  $x = -1\text{m}$  and  $x = 10\text{cm}$ .
8. A circular pool in a zoo has a diameter of 40 meters, and the spectators are separated from the pool by a fence 2 meters away from the edge of the pool. How much area is between the pool and the fence?
9. Simplify
  - (a)  $10^7 \cdot 10^3$
  - (b)  $(10^7)^3$
  - (c)  $\frac{10^{-7} x^8}{10^{-8} x^9}$
  - (d)  $(2x^3)^8$
10. Expand and simplify the expression  $(x + 4 \cdot 10^3)^2$ . Express all numerical values in scientific notation.
11. Simplify  $(x^{-1} + y^{-1})^{-1}$
12. Compute and simplify
  - (a)  $\frac{45 \text{ km}}{60 \text{ m/sec}}$
  - (b)  $\frac{5000 \text{ L/sec}}{25 \text{ m}^2}$
  - (c)  $\frac{500 \text{ cal}}{20 \text{ Watt}}$
13. Compute and simplify. Work using the pertinent formulas and confirm your results on a calculator.
  - (a)  $16^{3/4}$
  - (b)  $0.25^{3/2}$
  - (c)  $125^{-2/3}$
  - (d)  $\left(\frac{4}{9}\right)^{-3/2}$
  - (e)  $225^{-0.5}$
  - (f)  $0.000,320^{-1.2}$
14. A population has alleles A and B. Tests show that 80% of all gametes have genotype AA. What percentage of the population has genotype BB if the population is in Hardy-Weinberg equilibrium?
15. A species has a gene with alleles A and B. In a specific population of 4,670 you count 3,781 individuals with genotype AA. How many have genotype AB and how many have genotype BB, if the population is in Hardy Weinberg equilibrium?
16. There are 22 black mice (BB), 34 grey mice (Bb) and 15 white mice (bb) in a habitat. After the next breeding season the population will reach 120 mice. Given that the population is in Hardy-Weinberg equilibrium, estimate the number of mice for each phenotype.

## Answers

1. (a)  $12 + 3x$   
 (b)  $-90 - 15x$   
 (c)  $-40 + 4x$   
 (d)  $4 - x^2$   
 (e)  $9x - 3x^2 + 3xy$   
 (f)  $25 + 20x + 4x^2$   
 (g)  $x^2 - 6x + 9$   
 (h)  $x^3 - 4x^2 + 4x$
2. (a)  $7(3 - x)$   
 (b)  $x(2y - x)$   
 (c)  $(12 - x)(12 + x)$   
 (d)  $(2x - 3)^2$
3. (a)  $\frac{7}{12}$   
 (b)  $\frac{x^2 + 1}{x}$   
 (c)  $3x$   
 (d)  $\frac{4x}{(x - 1)(x + 1)} = \frac{4x}{x^2 - 1}$   
 (e)  $\frac{3x}{4}$   
 (f)  $\frac{x^3}{48}$
4. \$ 76.97
5. 2599
6. 29,440
7.  $\pi(25 + x)^2 \text{ m}^2$   
 $= \pi(625 + 50x + x^2) \text{ m}^2$   
 $729\pi \text{ m}^2 = 2290 \text{ m}^2,$   
 $576\pi \text{ m}^2 = 1,810 \text{ m}^2$  and  
 $630.01\pi \text{ m}^2 = 1,979 \text{ m}^2$
8.  $264 \text{ m}^2$
9. (a)  $10^{10}$   
 (b)  $10^{21}$   
 (c)  $\frac{10}{x}$   
 (d)  $256x^{24}$
10.  $x^2 + 8 \cdot 10^3x + 1.6 \cdot 10^7$
11.  $\frac{xy}{x + y}$
12. (a) 750 seconds  
 (b) 0.2 m/sec  
 (c) 104.6 seconds
13. (a) 8  
 (b) 0.125  
 (c) 0.04  
 (d)  $\frac{27}{8}$   
 (e)  $\frac{1}{15}$   
 (f) 15,625
14. 1.1%
15. 842 have genotype AB, 47 have genotype BB.
16. 36.2 black, 59.4 grey and 24.4 white (round to whole numbers).

## 6 Graphs

Graphs are a splendid way to summarize data, and to present them visually. There are numerous ways to represent biological data. In this section we will focus on the mathematical side of graphing for scatter plots and functions; pie charts, frequency histograms or bar charts will not be considered. We look primarily at continuous variables (as opposed to discrete or qualitative variables). We will use mathematical examples, as well as illustrations from population growth, allometry, medical data and performance in college.

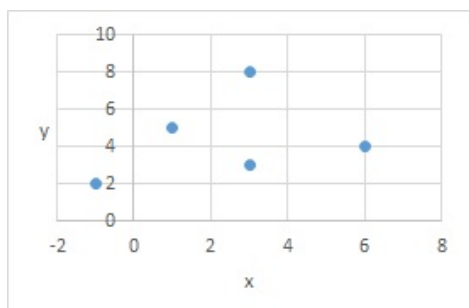
### 6.1 Scatter Plots

When you graph data from a two-column table, you interpret the numbers as respective  $x$  and  $y$  coordinates, and then you plot the points. In this fashion you can translate numerical data into points in a graph.

**Example:**

$x$	$y$
1	5
3	8
3	3
6	4
-1	2

The figure shows the resulting graph obtained in EXCEL.



Each of the dots represents one pair on the list. There is no need to list the data in a particular

order, and we do not connect dots.

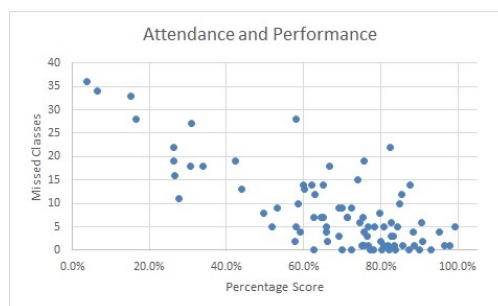
**Example:** Eight second graders had a physical. Here are the results.

Name	Height (cm)	Weight (kg)
Allie	115	20.1
Brian	125	23.3
Caleb	122	24.8
Dante	130	25.2
Emma	119	24.7
Frank	122	23.0
Garth	116	22.1
Haley	121	22.3

Although not necessary, the names have been included in the graph in order to emphasize that in a scatter plot each data pair becomes a single point in the graph.



**Example:** Scatter points look more impressive for larger data sets. The figure below relates student's grades to their attendance records.



The horizontal axis shows the percentage score, and the vertical axis denotes the number of missed classes <sup>5</sup> for students in a MATH 119 class. Students in the upper left corner quit coming to class, gave up on homework and did not take exams. None of the students with a perfect attendance record failed the class (one struggled). Attaching names would break confidentiality, and it would clutter up the graph without giving new meaningful information.

Scatter plots can also be done on graphing calculators using the "LIST" environment. But data entry is a daunting task, and the plots usually do not look very appealing. A computer is a much better tool for this task.

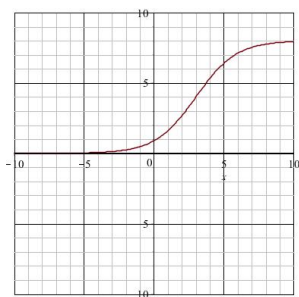
## 6.2 Function-Style Graphs

This is the type of graph, which you obtain on a calculator. It requires that you type in a formula for  $y$ , and let the calculator do the rest.

Take, for example, the problem

$$y = \frac{2^x}{1 + 2^{x-3}}$$

A graphing calculator will show a picture like the one given in the figure.



Typically, the  $x$  values and the  $y$  values will range from  $-10$  to  $10$  on a calculator, but you

<sup>5</sup>This is an example of a finite discrete variable; it can take on the values  $0, 1, 2 \dots 42$  only.

can change that using the WINDOW button. If you switch to the TABLE setting, the display will give you data for selected  $x$  values.

As it was the case for scatter plots, pairs of  $x$  and  $y$  are being displayed, but there are two major differences:

- We have infinitely many  $x$  values, at least in theory<sup>6</sup>.
- For each  $x$  value there is only one  $y$ , which results from the formula<sup>7</sup>. In fact, the formula may fail to produce an answer, if for example you are trying to divide by zero, or take the square root of a negative number.

This graphing method is tied to the notion of functions. A function is an input-output relationship: To each (admissible) input  $x$ , we assign exactly one output  $y$ . The letter  $f$  is frequently used to denote a function, and we write

$$y = f(x)$$

pronounced "f of x".  $x$  is the input, and  $y = f(x)$  is the output. A more detailed discussion of functions is presented in Chapter 11.

Many examples for function-style graphs in biology use time  $t$  as the input. The outputs could be things like populations, temperatures, oxygen concentrations, and so on, measured at time  $t$ . But time is not the only conceivable variable. One could display photosynthetic energy production  $y$  as a function of the size of a leaf  $x$ , or the density of algae  $y$  as a function the depth  $x$  below the water surface, and so on.

<sup>6</sup>A calculator, or any other graphing device for that matter, will only display a finite number of points, and connect them by line segments. But the points are so close together that they appear like a smooth curve.

<sup>7</sup>In the 2nd grader example Caleb and Frank are both 122 cm tall, but Caleb weighs more. This results in different  $y$  values for the same  $x$ -coordinate.

Graphs can also be used to confirm the correctness of an algebraic calculation. For example, in the last section we found that (simplify the expression ...)

$$\frac{x - \frac{1}{x}}{x + 1} = 1 - \frac{1}{x}$$

If we graph both sides of this equation on a calculator, we only see one graph, the reason being, that for any  $x$ -value, the values on the right and on the left are equal<sup>8</sup>. After all, simplifying means that we don't change the value of an expression, we just make it easier to handle.

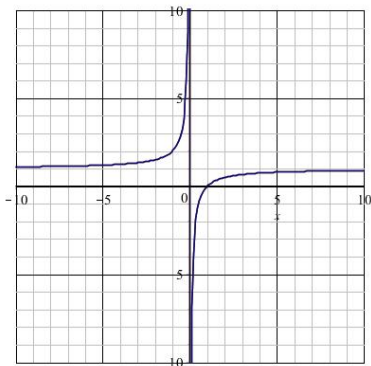


Figure 11:  $\frac{x - \frac{1}{x}}{x + 1} = 1 - \frac{1}{x}$

## 6.3 Watch the Scales

### 6.3.1 Log Plots

We begin with an example. The U.S. census data are given in Figure 12.

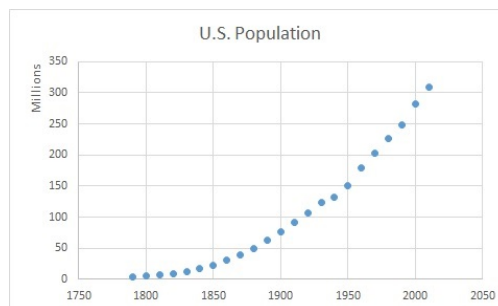
The scatter plot is shown below. It has the typical shape of an exponential growth. A dis-

<sup>8</sup>Maybe you have noticed a minor technical difference. If we take  $x = -1$ , the value on the left is undefined, but the value on the right is 2. A further investigation would take us right into Calculus.

Year	U. S. Population
1790	3,929,214
1800	5,236,631
1810	7,239,881
1820	9,638,453
1830	12,866,020
1840	17,069,453
1850	23,191,876
1860	31,443,321
1870	38,558,371
1880	49,371,340
1890	62,979,766
1900	76,212,168
1910	92,228,531
1920	106,021,568
1930	123,202,660
1940	132,165,129
1950	151,325,798
1960	179,323,175
1970	203,211,926
1980	226,545,805
1990	248,709,873
2000	281,421,906
2010	308,745,538

Figure 12: U.S. Historical Data

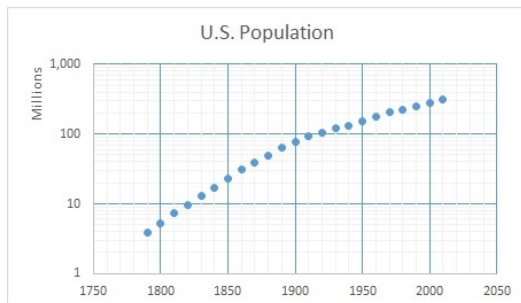
advantage of this display is that the large (recent) population figures dominate the graph, and we can barely make out differences for the data from the early 1800s.



In this situation you will often see a display with a logarithmic scale on the  $y$ -axis. Here



the powers of 10 are equally spaced, and adjustments are made in between.



As we go down on the  $y$ -axis, the values get closer and closer to zero (0.01, 0.001, 0.0001 ...), but they are never equal to zero. It is impossible to represent zero, or negative numbers on such a scale. A glimpse of the logarithmic scale is shown in Figure 13 below. The distance between 1 and 10 is the same as the distance between 2 and 20, or between 0.8 and 8 on this scale.

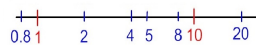


Figure 13: Logarithmic Scale

### 6.3.2 Loglog Plots

Graphs with logarithmic scales on both axes are called loglog plots, and they are extremely useful in applications.

**Example:** You investigate how the diversity of grasses on a meadow changes with area. On the smallest plot of 50 cm  $\times$  50 cm you find just two species. As you increase the lot size you detect more and more species. On the largest plot (50 m  $\times$  50 m) you count 31 different species. The data are summarized in the

table below.

Lot size( $m^2$ )	No. of Species
0.25	2
1	4
12	7
50	11
120	13
600	21
2,500	31

In a regular scatter plot we see that a fairly nice curve emerges, but the graph is dominated by the largest lot, and the data for the smaller lots are cluttered near the vertical axis.

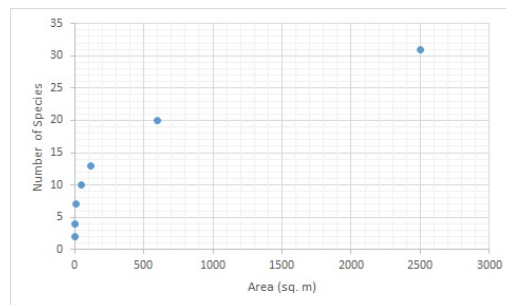


Figure 14: Area Species Curve

By changing to a loglog scale we can show more detail on both ends of the scale, small and large. Moreover, a fairly linear graph emerges, which caught the attention of biologists.

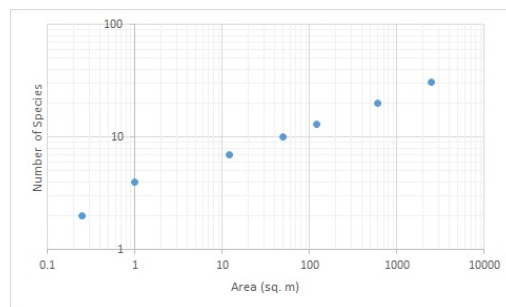


Figure 15: Area Species Curve in Loglog Scale

## 6.4 Worked Problems

EXCEL or other graphing software are very useful for these exercises.

1. **Temperature Graph.** Answer the questions based on the graph.

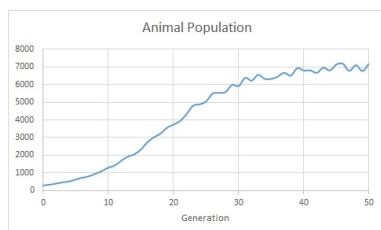


- (a) What were the high and the low temperatures in Radford on February 25?
- (b) When were the temperatures above freezing?
- (c) When did the temperatures change most rapidly?

Solution:

- (a) The low at 21° F between 6 am and 7 am; the high was 34° F at 4 pm.
- (b) The temperature remained above freezing between 1 pm and about 5:30 pm.
- (c) The fastest increase occurred between 8 am and 9 am.

2. **Animal Population.** The graph of a hypothetical animal population is shown below.



- (a) How many animals were present at time zero?
- (b) After how many generations will the population reach 2,500 individuals?
- (c) At what number will the population level off (carrying capacity)?

Solution:

- (a) The original population is about 300.
  - (b) The population breaks the 2,500 mark after about 15 generations.
  - (c) The population levels off at about 7,000.
3. Consider the curve defined by the equation

$$y = x^4 - 6x^2 + 5x + 6$$

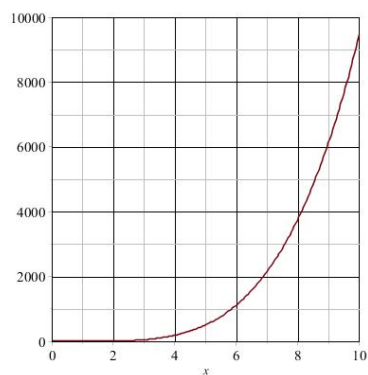
- (a) Complete the table below

$x$	$y$
0	
1	
2	
5	
10	

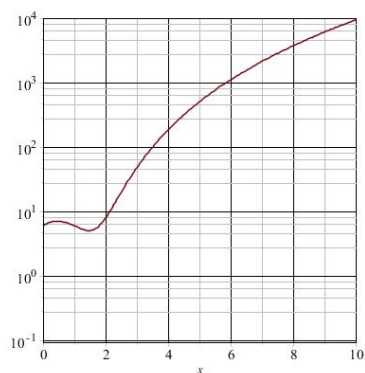
- (b) Graph the curve on a regular scale for  $x$  between 0 and 10.
- (c) Graph the function for  $x$  between 0 and 10 on a log scale.
- (d) Graph the function for  $x$  between 0.1 and 10 in a loglog plot.

Solutions:

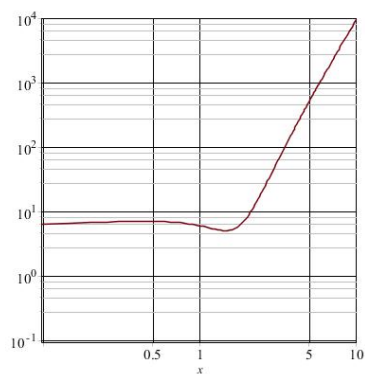
$x$	$y$
0	6
1	6
2	8
5	506
10	9,456



(b)



(c)



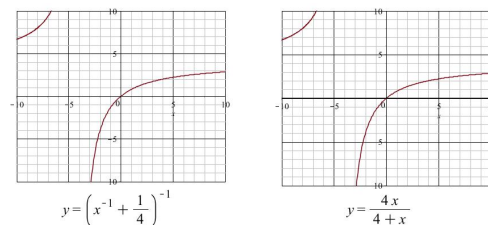
(d)

4. Simplify the expression  $\left(x^{-1} + \frac{1}{4}\right)^{-1}$  and confirm the result graphically.

Solution:

$$\begin{aligned}\left(x^{-1} + \frac{1}{4}\right)^{-1} &= \frac{1}{x^{-1} + \frac{1}{4}} \cdot \frac{4x}{4x} \\ &= \frac{4x}{4 + x}\end{aligned}$$

The graphs of  $\left(x^{-1} + \frac{1}{4}\right)^{-1}$  and  $\frac{4x}{4 + x}$  are shown below.



## 6.5 Exercises

1. Construct a scatter plot for the data below.

$x$	$y$
4	7
2	6
8	5
3	3
0	4
7	2

2. The Life Table for Belding's Ground Squirrels at Tioga Pass in Nevada (see Campbell, p. 1173) contains the following data. These are the number of surviving animals of a cohort in the beginning of the year.

Age	Females	Males
0 – 1	337	349
1 – 2	252	248
2 – 3	127	108
3 – 4	67	34
4 – 5	35	11
5 – 6	19	2
6 – 7	9	
7 – 8	5	
8 – 9	4	
9 – 10	1	

- (a) Graph the data for females and males in a common plot with regular scaling.

- (b) Repeat part (a) with a logarithmic scale on the  $y$ -axis.
3. The table below contains the heart rate at rest for selected mammals (Kong, NIU).

	Mass (g)	Pulse (beats/minute)
Mouse	25	670
Rat	200	420
Guinea pig	300	300
Rabbit	2,000	205
Small dog	5,000	120
Large dog	30,000	85
Man	70,000	72
Horse	450,000	38

Graph the data with a scatter plot on a loglog scale. It is not necessary to label the data points.

4. Let  $y = x^2 - 3x + 5$ . Complete the table and sketch the graph.

$x$	$y$
-1	
0	
1	
2	
3	
4	

5. Sketch the graph of

$$y = \frac{2^x}{2^x + 1}$$

for  $-4 \leq x \leq 4$ .

6. Sketch the graph of  $y = x^2 - 3x + 5$  for  $0.1 \leq x \leq 100$  on a loglog scale.
7. Sketch the graph of

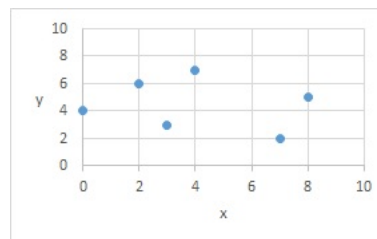
$$y = \frac{4^x + 3^x}{5^x + 2^x}$$

for  $-10 \leq x \leq 10$  on a logarithmic scale.

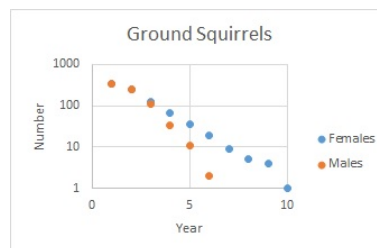
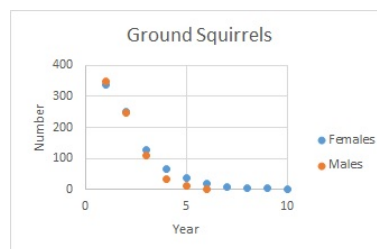
8. Factor the expression  $x^4 - 16$  and confirm the result graphically.

## Answers

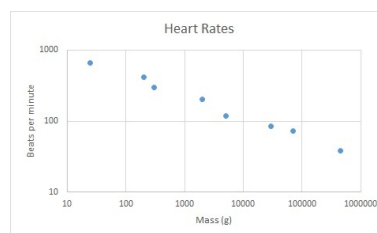
1.



2.

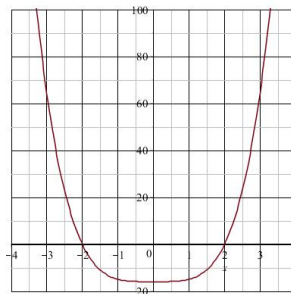
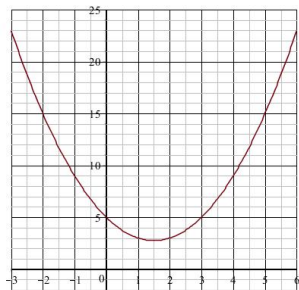


3.

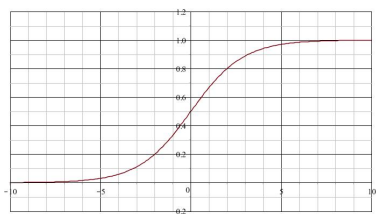


4.

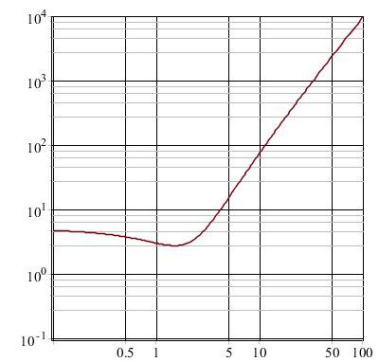
$x$	$y$
-1	9
0	5
1	3
2	3
3	5
4	9



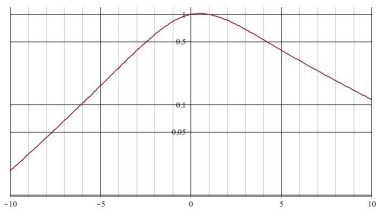
5.



6.



7.



8.  $(x - 2)(x + 2)(x^2 + 4)$

## 7 Solving Equations

Equations are everywhere. In this section we look at the basics of solving equations, and the techniques discussed here will be applied in the remainder of this workbook in a variety of biological applications.

### 7.1 Equations and Solutions

In the last section we dealt with expressions. They can be evaluated, simplified, graphed, expanded, factored, but they cannot be *solved*, because they lack an equality symbol.

An *equation* is a statement which claims that *two expressions* are equal to each other. For example, in

$$x^2 + 4 = 5x$$

we are looking for values of the variable  $x$  such that the expressions  $x^2 + 4$  and  $5x$  have the same value.  $x^2 + 4$  and  $5x$  taken by themselves are just *expressions*; we created an *equation* by connecting them with an “=”.

A *solution* of an equation is a value of the variable, such that equality is achieved. For instance, the equation

$$x^2 + 4 = 5x$$

has solutions  $x = 1$  and  $x = 4$  because

$$\begin{array}{lll} x = 1 & : & 1^2 + 4 = 5 \quad \text{and} \quad 5 \cdot 1 = 5 \\ x = 4 & : & 4^2 + 4 = 20 \quad \text{and} \quad 5 \cdot 4 = 20 \end{array}$$

$x = 3$  is not a solution, because  $3^2 + 4 = 13$ , while  $5 \cdot 3 = 15$ .

To *solve* an equation means to find *all possible solutions*.

*How many solutions* does an equation have? The answer depends on the equation. It can have exactly one solution, it can have several

solutions, it can have infinitely many solutions, or it may not have a solution at all.

**Examples:**

$$\begin{array}{lll} 4x + 7 & = & x + 22 \quad x = 5 \\ x^2 + 4 & = & 5x \quad x = 1 \text{ and } x = 4 \\ \sqrt{x^2} & = & x \quad \text{all } x \geq 0 \\ 4x + 7 & = & 4x - 12 \quad \text{no solution} \end{array}$$

### 7.2 Solution Strategies

Before you get overwhelmed by technicalities, keep the overall goal in mind: You want to find all  $x$  so that the equality holds true. Sometimes you can tell the answer by inspection (or by a lucky guess), sometimes there is a formula to find the solutions, sometimes it may take many hours pursuing a systematic approach.

In this section we will first look at the pencil and paper method (algebra), then we look for graphical solutions, and we conclude with the special case of quadratic equations.

#### 7.2.1 Algebraic Methods

When you use algebra, you try to isolate the variable. There are two operations which do not change the solution set:

1. add (subtract) the same quantity to both sides of the equation,
2. multiply (divide) both sides of the equation by a non-zero quantity.

It is left to you to decide which steps should be taken to solve the problem efficiently.

**Example:**

$$\frac{1}{4}x^2 = x + 3$$

First, we get rid of the fraction and multiply both sides by 4:

$$x^2 = 4x + 12$$

Now we subtract  $4x + 12$  from both sides

$$x^2 - 4x - 12 = 0$$

At this point factoring comes in handy

$$(x - 6)(x + 2) = 0$$

From here we conclude that either  $x = 6$  or that  $x = -2$ , since these are the only two choices for  $x$  which make the expression on the left equal to zero.

It is usually a good idea to double-check and substitute the values for  $x$  into the original equation. When  $x = 6$  we have

$$\frac{6^2}{4} = 9 = 6 + 3$$

and both sides equal 9. When we substitute  $x = -2$  we find that

$$\frac{(-2)^2}{4} = 1 = -2 + 3$$

and both sides equal 1.

The **Zero-Product-Principle** is a powerful tool to solve equations. It states that if the product of numbers is zero, then one of the factors must be zero.

**Example:** If

$$x(x - 1)(x + 2)(x - 3)^2 = 0$$

then  $x = 0$ , or  $x = 1$ , or  $x = -2$  or  $x = 3$ , and we see that this equation has four solutions.

It is essential that one side of the equation is zero. We cannot tell much about the solution set from the equation

$$x(x - 1)(x + 2)(x - 3)^2 = 8$$

because the value on the right is not zero.

### 7.2.2 Discussion of Pitfalls

In this section we look at typical examples where after careful calculations we end up with answers which do not satisfy the original equations, or where we miss some of the solutions. Squaring both sides, or multiplication or division by zero are often at the heart of the problem.

**Squaring both Sides.** If two quantities are equal, then their squares must be the same. True, but two numbers which are not equal can have the same square, as in  $2^2 = 4 = (-2)^2$ , and this may be at the source of the problem.

**Example:**

$$\sqrt{3x - 5} = 5$$

When we take the square on both sides, we find that

$$3x - 5 = 25$$

and therefore  $3x = 30$  and  $x = 10$ . Substitution confirms that

$$\sqrt{3 \cdot 10 - 5} = \sqrt{25} = 5$$

No problem here.

**Example:** Here we show how squaring both sides might introduce extraneous solutions. We consider the equation

$$x - 1 = 3$$

Obviously,  $x = 4$  is the only solution, and squaring both sides is not necessary. But, for the sake of making a point, let's say we square both sides, and then take it from there. We obtain

$$x - 1 = 3$$

$$(x - 1)^2 = 9$$

$$x^2 - 2x + 1 = 9$$

$$x^2 - 2x - 9 = 0$$

$$(x - 4)(x + 2) = 0$$

with solutions  $x = 4$  and  $x = -2$ . The first answer is the original solution.  $x = -2$  became a legitimate solution after we took the square on both sides. This illustrates the danger of squaring both sides, and it is good practice to confirm your final results for the original equation.

**Multiplication or Division by Zero.** Nobody in their right mind would do this. Division by zero is generally prohibited, and multiplication by zero makes both sides of the equation zero, and we lose all information. The crux is that multiplication or division by zero are usually hidden by the use of variables.

**Example:** Consider this string of calculations

$$\begin{aligned}x^2 - 4 &= x - 2 \\(x - 2)(x + 2) &= x - 2 \\x + 2 &= 1 \\x &= -1\end{aligned}$$

$x = -1$  is a solution to the original equation, because  $(-1)^2 - 4 = 1 - 4 = -3 = (-1) - 2$ , and  $x = -1$  makes both sides equal  $-3$ , no doubt about that.

But  $x = 2$  works as well in the original problem ( $4 - 4 = 0 = 2 - 2$  and both sides are 0). So, where did  $x = 2$  disappear? In the step from the second to the third line we divided by the quantity  $(x - 2)$ . But when  $x = 2$  we have  $(x - 2) = 0$ , and the division by  $(x - 2)$  knocked out the viable solution  $x = 2$ .

**Example:** This example is even more obvious:

$$\begin{aligned}x^2 &= 4x \\x &= 4\end{aligned}$$

The original equation has solutions  $x = 4$  and  $x = 0$ . The latter disappeared as an option after division by  $x$ .

**Example:** In this example we multiply both sides by  $x - 2$  in order to get rid of the fractions, and then take the usual simplification steps.

$$\begin{aligned}x + \frac{6}{x - 2} &= \frac{3x}{x - 2} \\ \frac{x^2 - 2x + 6}{x - 2} &= \frac{3x}{x - 2} \\ x^2 - 2x + 6 &= 3x \\ x^2 - 5x + 6 &= 0 \\ (x - 2)(x - 3) &= 0\end{aligned}$$

The solutions are solutions  $x = 3$  and  $x = 2$ .

$x = 3$  solves the original equation, because

$$3 + \frac{6}{1} = 9 = \frac{3 \cdot 3}{1}$$

but  $x = 2$  is not a solution, because the original equation is not defined (not meaningful) when  $x = 2$ . It became a viable answer after we multiplied both sides by  $(x - 2)$ . Here multiplication by zero was disguised as the factor  $(x - 2)$ .

### 7.2.3 Graphical Solutions

Solving equations graphically is always an option, no matter how hard or easy the problem might be! Efficient use of your graphing calculator will yield useful numerical results!

In the example

$$x^2 + 4 = 5x$$

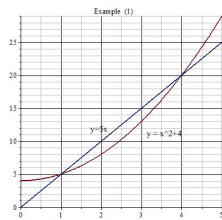
we are looking for values  $x$  so that both sides of the equation have the same value. The table shows values on both sides for different choices of  $x$ :

$x$	$x^2 + 4$	$5x$
0	4	0
<b>1</b>	<b>5</b>	<b>5</b>
2	8	10
3	13	15
<b>4</b>	<b>20</b>	<b>20</b>
5	29	25



and it is easy to see that both sides match when  $x = 1$  and when  $x = 4$ .

Instead of setting up tables, we can have a calculator graph the two expressions. We are now looking for choices of  $x$  with identical  $y$ -values. This means that we are interested in the points of intersection of the graphs.



From the graph we see that the intersections occur at the points (1, 5) and (4, 20). The  $x$ -coordinate is the desired solution. The  $y$  coordinate is a byproduct, as it shows the matching value of the expressions on either side.

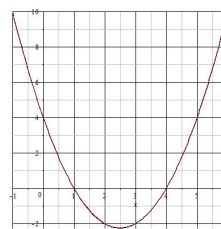
In textbook pencil and paper problems the solutions often work out nicely as whole numbers or simple fractions, but in most realistic problems this is not the case. The calculator is not afraid of messy numbers, and most graphing calculators have a way to identify points of directly intersections. Look for a CALC menu under graphing, select "intersect", and be prepared to answer a few questions (which curves are involved, what is the interval of interest). Detailed instructions depend on the brand of the calculator.

As an alternative to looking for intersection points you can also rewrite the equation so that one side is equal to zero. Now only one expression is involved, and you are looking for the  $x$ -intercepts of the curve.

For the problem  $x^2 + 4 = 5x$  you can switch to

$$x^2 + 4 - 5x = 0$$

and the graph of  $y = x^2 + 4 - 5x$  shows  $x$  intercepts at  $x = 1$  and  $x = 4$ .



Again, there is a quick way to do this on your calculator. Look for things like "zero" or "root" on a graphing menu.

## 7.2.4 Quadratic Equations

Mathematicians classify equations into various types. Equations of the form  $ax = b$  are called *linear*, and the solution is  $x = b/a$ , provided that  $a \neq 0$ .

Increasing the power of  $x$  results in a *quadratic equation*, which takes the general form

$$ax^2 + bx + c = 0$$

It is understood that  $a \neq 0$ , otherwise the equation would be linear. Quadratic equations can have two, one or no solutions.

There are three common ways to solve a quadratic equation.

1. **Factoring.** This is a fabulous short-cut! But it does not always work. For instance,

$$\begin{aligned} x^2 - 5x + 4 &= 0 \\ (x - 1)(x - 4) &= 0 \end{aligned}$$

can be worked by factoring and the solutions are  $x = 1$  and  $x = 4$ . Done!

The equation

$$x^2 - 4x + 1 = 0$$

does not factor easily, but it still has solutions (see below). Just because we cannot factor an expression doesn't mean that there are no solutions.

2. **Complete the Square.** This technique is not popular with students. But in the simple problem

$$x^2 = 16$$

factoring is not required. We have a perfect square already, and the solutions are  $x = \pm 4$ .

In the problem

$$(x - 3)^2 = 16$$

we have a perfect square again, and now the solutions are  $x - 3 = \pm 4$ . This implies that

$$x = 3 \pm 4,$$

and the solutions are  $x = 7$  or  $x = -1$ .

If the number on the right is not a square number, square roots must be used. For example,

$$(x - 3)^2 = 5$$

has solutions  $x = 3 + \sqrt{5} = 5.236$  or  $x = 3 - \sqrt{5} = 0.764$ .

The beauty of this technique lies in the fact that we can tell the number of solutions by the sign of the number on the right. For example

$$\begin{aligned}(x - 3)^2 &= 16 \\ (x - 3)^2 &= 0 \\ (x - 3)^2 &= -5\end{aligned}$$

The first equation has two solutions, as we saw above. The second equation has the solution  $x = 3$  only, and the last equation does not have any solutions, because squares cannot be negative.

What makes the method so unpopular, is that we have to rewrite the equation in

order for perfect squares to emerge (complete the square), for instance

$$\begin{aligned}x^2 - 4x + 1 &= 0 \\ x^2 - 4x + 4 &= 3 \\ (x - 2)^2 &= 3\end{aligned}$$

and the solutions are  $x = 2 - \sqrt{3}$  and  $x = 2 + \sqrt{3}$ .

3. **Quadratic Formula.** Fortunately, it is possible to solve the quadratic equation for any combination of the parameters  $a$ ,  $b$  and  $c$  by completing the square, and the final result is summarized in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term  $b^2 - 4ac$  is called the discriminant. If  $b^2 - 4ac > 0$ , the equation has two distinct solutions, if  $b^2 - 4ac = 0$  it has just one solution (namely  $x = -\frac{b}{2a}$ ), and if  $b^2 - 4ac < 0$  solutions cannot be found because we cannot take the square root of a negative number.

**Example:**  $x^2 - 4x + 1 = 0$

Here  $a = 1$ ,  $b = -4$  and  $c = 1$ , the discriminant is  $(-4)^2 - 4 = 16 - 4 = 12 > 0$ , and the solutions are

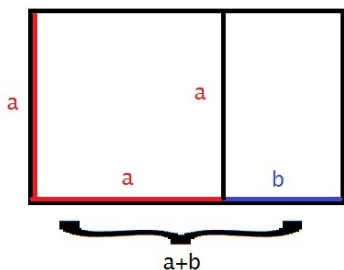
$$x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

In the last step we used that  $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$ . Now the fraction has a common factor of 2, and after cancelation we arrive at the final result.

Most of the time, equations are not served to us on a silver platter (standard form), and it is necessary to rewrite and to simplify them so that the formula can be applied. Examples

are given in the next section. Keep in mind that solving equations graphically is always an option, especially when the algebra gets out of control.

We close this part with an application: The **Golden Ratio** is observed in nature in plenty of forms. Just search the internet for "golden ratio in nature", and you will find a variety of applications, among them the ratio of the bones in our fingers or arms, or proportions of facial features, or the ratio of sections in bees (head, thorax, abdomen), and many more. The golden ratio also shows up in art and architecture.



Interestingly, the Golden Ratio is the solution of a quadratic equation. If the sides of a rectangle are denoted by  $a$  and  $b$ , then we have a golden rectangle, if  $a$  is to  $b$  what  $a+b$  is to  $a$ . We denote the golden ratio by  $\varphi$ , and put the information into equation form and to obtain

$$\varphi = \frac{a}{b} = \frac{a+b}{a}$$

But then

$$\begin{aligned}\varphi &= \frac{a+b}{a} = 1 + \frac{b}{a} \\ &= 1 + \frac{1}{\varphi}\end{aligned}$$

and multiplication by  $\varphi$  on both sides results in

$$\begin{aligned}\varphi^2 &= \varphi + 1 \\ \varphi^2 - \varphi - 1 &= 0\end{aligned}$$

Using the quadratic formula we see that

$$\varphi = \frac{1 \pm \sqrt{5}}{2}$$

The golden ratio is usually associated with the larger number

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.618,034$$

Notice, that the smaller solution is

$$\frac{1 - \sqrt{5}}{2} = -0.618,034 = 1 - \varphi$$

### 7.3 Worked Problems

1. Solve the given equation

- (a)  $\frac{x}{3} - 2 = 4$
- (b)  $\frac{x}{x-3} = \frac{3}{2}$
- (c)  $\sqrt{3x-2} = 5$
- (d)  $x(x-1) = (x-2)(x+3)$

Solutions:

(a) This is straightforward

$$\begin{aligned}\frac{x}{3} - 2 &= 4 \\ \frac{x}{3} &= 6 \\ x &= 18\end{aligned}$$

(b) First we multiply both sides by  $2(x-3)$ , which gets rid of the fractions. The rest is routine.

$$\begin{aligned}\frac{x}{x-3} &= \frac{3}{2} \\ 2x &= 3(x-3) = 3x-9 \\ 9 &= 3x-2x = x\end{aligned}$$

and we are done. The solution is  $x = 9$ . Substitution into the original equation confirms the result:

$$\frac{9}{9-3} = \frac{9}{6} = \frac{3}{2}$$

(c) Squaring both sides will get us started:

$$\begin{aligned}\sqrt{3x-2} &= 5 \\ 3x-2 &= 25 \\ 3x &= 27 \\ x &= 9\end{aligned}$$

Substitution into the original equation confirms the result:

$$\sqrt{3 \cdot 9 - 2} = \sqrt{27 - 2} = \sqrt{25} = 5$$

(d) We expand both sides first, and after subtracting  $x^2$  from both sides, we end up with a linear equation.

$$\begin{aligned}x(x-1) &= (x-2)(x+3) \\ x^2 - x &= x^2 + x - 6 \\ 6 &= 2x \\ x &= 3\end{aligned}$$

Check:

$$3 \cdot 2 = 6 = 1 \cdot 6$$

2. Solve the equations

- (a)  $2x^2 - 5x + 3 = 0$
- (b)  $(x-1)^2 - 9 = 0$
- (c)  $x^2 - 4x + 6 = 0$
- (d)  $x + \frac{1}{x} = 5$

Solutions:

- (a) Factoring looks challenging, but the quadratic formula will do the job. We have  $a = 2$ ,  $b = -5$  and  $c = 3$ . The discriminant is

$$b^2 - 4ac = (-5)^2 - 4 \cdot 2 \cdot 3 = 1$$

and thus

$$x = \frac{5 \pm \sqrt{1}}{4}$$

The solutions are

$$\begin{aligned}x &= \frac{5+1}{4} = \frac{6}{4} = 1.5 \\ x &= \frac{5-1}{4} = 1\end{aligned}$$

By the way, once you have the solutions, factoring is easy

$$2x^2 - 5x + 3 = (x-1)(2x-3)$$

- (b) We take advantage of the perfect square (do not expand!):

$$\begin{aligned}(x-1)^2 - 9 &= 0 \\ (x-1)^2 &= 9 \\ x-1 &= \pm 3 \\ x &= 1 \pm 3\end{aligned}$$

and the solutions are  $x = 4$  and  $x = -2$ .

- (c) This quadratic equation does not have a solution, because the discriminant is

$$\begin{aligned}b^2 - 4ac \\ = (-4)^2 - 4 \cdot 1 \cdot 6 &= -8 < 0\end{aligned}$$

- (d) Here we have to rearrange the equation before we can apply the quadratic formula.

$$\begin{aligned}x + \frac{1}{x} &= 5 \\x^2 + 1 &= 5x \\x^2 - 5x + 1 &= 0\end{aligned}$$

Therefore ( $a = 1$ ,  $b = -5$ ,  $c = 1$ )

$$x = \frac{5 \pm \sqrt{21}}{2}$$

The confirmation of this result is challenging. For  $x = \frac{5+\sqrt{21}}{2}$  we find that (multiply top and bottom by the "conjugate" - a routine step in algebra)

$$\begin{aligned}\frac{1}{x} &= \frac{2}{5 + \sqrt{21}} \cdot \frac{5 - \sqrt{21}}{5 - \sqrt{21}} \\&= \frac{2(5 - \sqrt{21})}{25 - 21} = \frac{5 - \sqrt{21}}{2}\end{aligned}$$

This shows that the two solutions are reciprocals of each other. Moreover,

$$\begin{aligned}x + \frac{1}{x} &= \frac{5 + \sqrt{21}}{2} + \frac{5 - \sqrt{21}}{2} \\&= \frac{5 + \sqrt{21} + 5 - \sqrt{21}}{2} = 5\end{aligned}$$

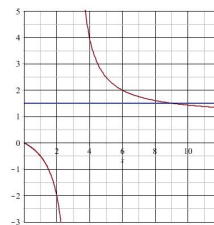
3. Solve the equations graphically. The equations are taken from Problems 1 and 2, and we can confirm our algebraic results by inspection of the resulting graphs.

- (a)  $\frac{x}{x-3} = \frac{3}{2}$   
 (b)  $2x^2 - 5x + 3 = 0$   
 (c)  $x^2 - 4x + 6 = 0$

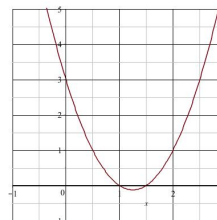
(d)  $x + \frac{1}{x} = 5$

Solutions

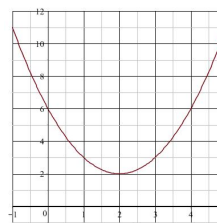
- (a) The graph of  $y = \frac{x}{x-3}$  has a vertical asymptote at  $x = 3$ , but most of all, it intersects with the horizontal line  $y = \frac{3}{2} = 1.5$  at the point where  $x = 3$ .



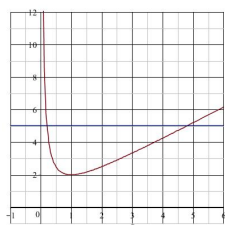
- (b) This quadratic equation has two solutions. One at  $x = 1$ , the other at  $x = 1.5$



- (c) This quadratic equation has no solution. The parabola  $y = x^2 - 4x + 6$  never crosses the  $x$ -axis.



- (d) Here we see that the curve  $y = x + \frac{1}{x}$  intersects with the line  $y = 5$  at two different locations. The first intersection occurs at  $x = \frac{5-\sqrt{21}}{2} = 0.209$ , and the second at  $x = \frac{5+\sqrt{21}}{2} = 4.701$ .

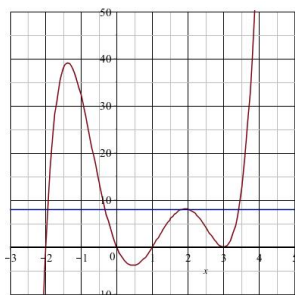


4. Solve  $x(x-1)(x+2)(x-3)^2 = 8$

Solution: This equation cannot be solved by factoring, because the value on the right is not zero. Expansion of the expression on the left leads to

$$x^5 - 5x^4 + x^3 + 21x^2 - 18x = 8$$

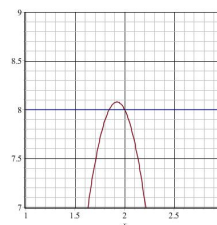
and there is no convenient method to solve such an equation explicitly. Graphing (or other numerical methods) are about the only recourse.



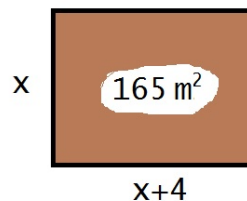
We find the solutions

$$\begin{aligned} x_1 &= -1.943 \\ x_2 &= -0.326 \\ x_3 &= 1.847 \\ x_4 &= 2.000 \\ x_5 &= 3.422 \end{aligned}$$

from the graph. The solutions near  $x = 2$  are clustered, but zooming shows that we have two solutions in this vicinity.



5. The length of a rectangular garden plot is four meters longer than its width. The area is 165 square meters. What are the dimensions?



Solution: Denote the width by  $x$ , then the length is  $x + 4$ , and the area becomes  $x(x + 4) = x^2 + 4x$ . Hence, the problem at hand is

$$x^2 + 4x = 165$$

We subtract 165 from both sides and then factor the result:

$$\begin{aligned} x^2 + 4x - 165 &= 0 \\ (x + 15)(x - 11) &= 0 \end{aligned}$$

with solutions  $x = 11$  and  $x = -15$ . We disregard the solution  $x = -15$ , as the sides of a rectangle cannot be negative. Thus the width becomes 11 m and the length is 15 m.

6. The height  $h$  of a projectile above the ground at time  $t$  is given by the equation

$$h = 80 + 64t - 16t^2$$

where  $h$  is measured in feet, and time is measured in seconds. After how many seconds will the object hit the ground?

Solution: These type of questions are commonplace in math books. In the feet-second system the term  $-16t^2$  accounts for the loss of height due to gravity.

In our case the projectile has height  $h = 80$  ft above the ground when  $t = 0$ , and after one second we are at  $h = 80 + 64 - 16 = 128$  ft, and so on. Hitting the ground means that  $h = 0$ . This results in the quadratic equation

$$80 + 64t - 16t^2 = 0$$

and the quadratic formula yields

$$\begin{aligned} t &= \frac{-64 \pm \sqrt{64^2 - 4 \cdot (-16) \cdot 80}}{-32} \\ &= 2 \pm \frac{\sqrt{9216}}{-32} = 2 \pm \frac{96}{-32} \\ &= 2 \pm 3 \end{aligned}$$

The solutions are  $t = 2 + 3 = 5$  seconds and  $t = 2 - 3 = -1$  sec. The negative solution is irrelevant (we don't go back in time), and the final result is that the projectile will hit the ground after five seconds.

## 7.4 Exercises

Solve for  $x$  in problems 1 - 20.

1.  $2x - 3 = 7$

2.  $\frac{2x-1}{5} = 5$

3.  $\frac{x+2}{x-1} = 4$

4.  $\frac{x+2}{x-1} = 1$

5.  $\sqrt{4-x} = 3$

6.  $(x-5)(x-1) = (x-4)(x-2)$

7.  $(x-5)(x-4) = (x-2)(x-1)$

8.  $\frac{x^2-4}{x+2} = x+2$

9.  $\frac{x^2-4}{x+2} = x-2$

10.  $\frac{x+2}{x-1} = \frac{x+3}{x-3}$

11.  $\sqrt{x+56} = 4 + \sqrt{x}$

12.  $\frac{1}{x} + 1 = \frac{3}{x}$

13.  $x^2 - 2x - 99 = 0$

14.  $x^2 + 25 = 10x$

15.  $2x^2 - x - 3 = 0$

16.  $x^2 - 2x - 2 = 0$

17.  $x^2 + 2x + 4 = 0$

18.  $(x-2)^2 - 4 = 0$

19.  $x + \frac{3}{x} = 4$

20.  $x + \sqrt{x} = 12$

21. The sides of a rectangle are in golden ratio. Find the length of the remaining side when

(a) the longer side is 55 cm,

(b) the short side is 55 cm.

22. The length of a rectangular garden is three meters longer than the width.

(a) What are the dimensions if the area is 108 square meters?

- (b) What are the dimensions if the area is 100 square meters?
23. The height  $h$  above ground measured of a baseball  $t$  seconds after it is hit is given by the formula

$$h = 4 + 40t - 16t^2$$

where  $h$  is measured in feet. After how many seconds will it hit the ground?

19.  $x = 1$  and  $x = 3$
20.  $x = 9$
21. (a)  $33.992 \text{ cm} \approx 34 \text{ cm}$   
(b)  $88,992 \text{ cm} \approx 89 \text{ cm}$
22. (a)  $12 \times 9$  (b)  $11.612 \times 8.612$
23. 2.6 seconds (2.5963)

### Answers

1.  $x = 5$
2.  $x = 13$
3.  $x = 2$
4. no solution
5.  $x = -5$
6. no solution
7.  $x = 3$
8. no solution
9. all  $x \neq -2$
10.  $x = -1$
11.  $x = 25$
12.  $x = 2$
13.  $x = 11$  and  $x = -9$
14.  $x = 5$
15.  $x = \frac{3}{2}$  and  $x = -1$
16.  $x = 1 \pm \sqrt{3}$ , that is,  
 $x = 2.732$  and  $x = -0.732$
17. no solution
18.  $x = 0$  and  $x = 4$



## 8 Ratios and Proportion

In this part covers ratios and proportion, and we will review percentages and related topics. Important biological applications include dilutions of chemicals, the mark and recapture method, as well problems from medicine, ecology and epidemiology.

### 8.1 Ratios

Ratios are typically the division of terms with the same units. The result is a number which does not have any physical units. Such a quantity is called *dimensionless*.

**Example:** The RU Factbook 2014 reports 411 biology majors and 75 mathematics major. The ratio of majors is

$$\frac{411 \text{ students}}{75 \text{ students}} = 5.48$$

The units in numerator and denominator are "students". They cancel, and the result is a number. This ratio is about 5.5, and we conclude that the ratio biology:math majors is 11:2 ( $\frac{11}{2} = 5.5$ ).

**Example (Circles):** For all circles, large or small, the ratio of the perimeter (circumference) to the diameter is the same.

Diameter	Perimeter	Ratio
4.2 in	13.2 in	3.143
16 miles	50 miles	3.125
78 mm	245 mm	3.141
2.3 m	7.2 m	3.130
55 $\mu\text{m}$	173 $\mu\text{m}$	3.145

The slight variations of the ratios are due to rounding. Using the formulas from geometry we find that

$$\frac{\text{perimeter}}{\text{diameter}} = \frac{2\pi r}{2r} = \pi$$

This is the definition of  $\pi$ . It is the ratio of the perimeter of a circle to its diameter.

**Example (Energy Payback Ratio):** This ratio relates the energy output of a power plant to the energy input. When 2,800,000 GJ (giga-joules) are required to produce 11,350,000 GJ by a natural gas powered plant, the payback ratio becomes

$$\frac{\text{energy output}}{\text{energy input}} = \frac{11,350,000 \text{ GJ}}{2,800,000 \text{ GJ}} = 4.053$$

Each unit of energy input results in a fourfold of energy output.

Incidentally, the data are based on a study by Meier and Kulcinski of the University of Wisconsin. The same authors report payback ratios of 11:1 for coal, 27:1 for fusion and 23:1 for wind energy.

### 8.2 Percent

Percents usually appear in conjunction with the ratio

$$\frac{\text{parts}}{\text{whole}}$$

The result will be a number between 0 and 1, and to make the number more user-friendly it is multiplied by 100.

The term *percent* goes back the the Latin *per centum*, and means by the hundred. We use the symbol % to express percents. Mathematically we have

$$\% = \frac{1}{100} = 0.01 \quad \text{and} \quad 100\% = 1$$

For example,

$$45\% = 45 \cdot \frac{1}{100} = \frac{45}{100} = 0.45$$

**Example:** A class has 20 female and 16 male students for a total of 36 students. In this case

the percentage of female students is computed from

$$\frac{20 \text{ female students}}{36 \text{ total students}} = \frac{20}{36} = \frac{5}{9}$$

$$= 0.55556 = 55.6\%$$

while the ratio becomes

$$\frac{20 \text{ female students}}{16 \text{ male students}} = \frac{20}{16} = \frac{5}{4}$$

that is, the ratio female:male students is 5:4.

**Example (HIV in Nigeria):** 3.1% of the adult population of Nigeria was infected with HIV in 2012 (CIA World Factbook).

When 3.1% of the population being infected, then 96.9% are not infected, and for the ratio we find that

$$\frac{\text{not infected}}{\text{infected}} = \frac{96.9\%}{3.1\%} = 31.258$$

Roughly speaking, we have one HIV infected person per 31 HIV-free people.

## 8.3 Percentlike Quantities

### 8.3.1 Parts per Thousand

When the reference is changed from one hundred to one thousand, we speak of parts per thousand, sometimes also called per mille. Common notations are

$$\text{‰} \quad \text{or} \quad \text{ppt}$$

**Example:** The United Nations reported in 2010 that the infant mortality<sup>9</sup> in Japan was 2.62 deaths per 1,000 life births. The infant mortality rate then becomes

$$\frac{2.62 \text{ deaths}}{1,000 \text{ births}} = 0.002,62 = 2.62\text{‰}$$

<sup>9</sup>Death of a child one year or younger.

In comparison, in 2012 the CDC found that the infant mortality rate in the United States was 5.98‰.

**Example:** The legal limit of a drivers blood alcohol concentration in Virginia is 0.08%. This is equivalent to 0.8‰.

Both examples show that changing from percent to per mille, the decimal place is shifted by one place.

### 8.3.2 Parts per Million, Parts per Billion

When quantities get really small, we resort to parts per million (ppm) or parts per billion (ppb).

**Example:** The abundance of hydrogen isotopes in the atmosphere are

$$\begin{array}{ll} {}^1\text{H} & 99.985\% \\ {}^2\text{H} & 0.015\% \end{array}$$

Thus, the abundance of  ${}^2\text{H}$  is

$$0.015\% = 0.15\text{‰} = 150 \text{ ppm}$$

$$\frac{0.015}{100} = \frac{0.15}{1,000} = \frac{150}{1,000,000}$$

that is, 150 of a million hydrogen atoms are the  ${}^2\text{H}$  isotope.

**Example:** Radford drinking water contains 1.52 ppm of chlorine (Radford City Drinking Water Report, 2013).

Examination of the fine print reveals that ppm is used as milligram per liter (mg/L), and it follows that one liter contains 1.52 mg of chlorine. Conversion to gallons yields

$$\frac{1.52 \text{ mg}}{\text{L}} \cdot \frac{3.785 \text{ L}}{\text{gal}} = \frac{5.75 \text{ mg}}{\text{gal}}$$

Using ppm as mg/L is a violation of the condition that things like percentages should be free of physical units. However, it is common practice to use the density of water

$$\rho = 1 \text{ kg/L} = 1,000 \text{ g/L}$$

when dealing with aqueous solutions. Then one ppm becomes one milligram per liter:

$$\frac{\rho}{1,000,000} = \frac{1000 \text{ g}}{1,000,000 \text{ L}} = \frac{\text{mg}}{\text{L}}$$

## 8.4 Percentage Change

Now we look at relative changes, expressed as a percentage. We hear things like "cost of living increased by 2%", or "India's tiger population has increased by 30%" (BBC News, 1/25/2015) on a daily basis. What does it mean, and how is it calculated?

In order to find a relative change we compute the quantity

$$\frac{\text{final value} - \text{initial value}}{\text{initial value}}$$

Multiplication of the result by 100% yields the *percentage change*.

**Example:** Bacteria colonies in a laboratory increased from 720 colonies per liter to 1260 colonies per liter. Here the relative change becomes

$$\frac{1260 - 720}{720} = \frac{540}{720} = 0.75 = 75\%$$

**Example:** India's tiger population has risen from 1,706 in 2011 to 2,226 in 2014. The percentage change is

$$\frac{2226 - 1706}{1706} = \frac{520}{1706} = 0.305 = 30.5\%$$

This is an increase of 30% over three years, which accounts for an annual growth of about 10%.

There are different ways to look at a relative change, or to calculate it. The quantity

$$\Delta \text{ value} = \text{final value} - \text{initial value}$$

is the absolute change. This quantity usually becomes more meaningful when we relate it to the beginning value.

**Example:** The population of the United States grew by 20.6 million during the years 2005-2013, while that of Norway increased only by 461 thousand during the same time span. Comparison to the actual populations in 2005 results in

$$\frac{20.6 \text{ million}}{295.5 \text{ million}} = 7.0\%$$

for the United States and in

$$\frac{0.461 \text{ million}}{4.623 \text{ million}} = 10.0\%$$

for Norway. This shows that in relative terms Norway's population outpaced that of the United States during 2005 - 2013 time span.

The relative growth formula can also be broken down into

$$\begin{aligned} r &= \frac{\text{final value} - \text{initial value}}{\text{initial value}} \\ &= \frac{\text{final value}}{\text{initial value}} - 1 = M - 1 \end{aligned}$$

The quantity

$$M = \frac{\text{final value}}{\text{initial value}} (= 1 + r)$$

is the multiplier (growth factor) and  $r$  is the relative growth rate. This will become a major theme in the study of exponential growth.

**Example:** We return to India's tigers. We have

$$M = \frac{2,226}{1,706} = 1.305$$

that is, the population grew by a factor of 1.305, and the relative growth is

$$r = M - 1 = 0.305 = 30.5\%$$

as before.

## 8.5 Proportion

Two quantities  $x$  and  $y$  are *proportional*, if their ratio remains constant, and we write

$$x \propto y$$

In this case we say that the quantities *vary directly* with each other.

A constant ratio also implies that

$$\frac{y}{x} = c$$

and thus,  $y = cx$ , where  $c$  is the constant of proportionality.

**Example:** Time and distance are proportional when you travel at a constant speed on a highway. If you spend twice as much time, you will go twice as far; if plan to travel only one third of the distance, it will take one third of the time. We have

$$\text{distance} \propto \text{time}$$

The constant of proportionality becomes

$$c = \frac{\text{distance}}{\text{time}}$$

which is the speed. We see that when  $x \propto y$ , where  $x$  and  $y$  have different physical units, then the units of  $c$  balance the equation.

**Example:** Maps use the principle of proportionality: The ratio of map distance to distance in reality always remains the same. As an example we look at the scale 1:200,000, which is widely used in aviation. This means that

$$\frac{\text{map distance}}{\text{actual distance}} = \frac{1}{200,000}$$

or equivalently,

$$\begin{aligned} & \text{actual distance} \\ &= 200,000 \times \text{map distance} \end{aligned}$$

On such a map 8 cm represent

$$200,000 \cdot 8 \text{ cm} = 1,600,000 \text{ cm} = 16 \text{ km}$$

in reality, or conversely, 600 m in reality are shown as

$$\frac{600 \text{ m}}{200,000} = 0.003 \text{ m} = 3 \text{ mm}$$

on this map.

**Example (Unit Conversion):** Blood sugar values can be measured in mmol/L (millimol per liter), but it customary to use mg/dL (milligram per deciliter<sup>10</sup>) readings.

The atomic mass of glucose  $C_6H_{12}O_6$  is 180 Da (12·6 Da for the six carbon atoms, 12 Da for hydrogen, and 16·6 Da from oxygen). Hence, one mol of glucose has mass 180 g. The values on the two scales will remain proportional and we find that

$$\begin{aligned} \frac{1 \text{ mmol}}{\text{L}} &= \frac{0.001 \text{ mol}}{10 \text{ dL}} \cdot \frac{180 \text{ g}}{\text{mol}} \\ &= \frac{0.018 \text{ g}}{\text{dL}} = 18 \frac{\text{mg}}{\text{dL}} \end{aligned}$$

Therefore, multiplication by 18 takes you from mmol/L to mg/dL. For instance, 7.5 mmol/L is equivalent to  $7.5 \cdot 18 = 135 \text{ mg/dL}$ .

## 8.6 Mark and Recapture

This is a technique which is used to estimate populations in the wild. It is also known as capture-recapture, as mark-release-recapture or as sight-resight. The basic idea is to capture a few animals and to mark them in a way so that they can be identified later. The animals

<sup>10</sup>One deciliter equals 0.1 liter.

are then released. A second capture follows at a later date. It is assumed that the percentage of tagged animals in the second catch is representative of the percentage of marked animals in the entire population.

We use the following notation

$N$	total population
$T$	tagged in the first catch
$n$	number of second catch
$t$	tagged in second catch

If we argue that percentages (ratios) remain identical we find that

$$\frac{t}{n} = \frac{T}{N}$$

One could also argue that  $t/T = n/N$ , but in any case it follows that

$$N = \frac{Tn}{t}$$

and we have a population estimate.

**Example:** In a study of birds on an island 71 birds are captured, tagged and released. Three days later 83 birds are caught, 23 of which are tagged from the first capture. Thus we have the values

$$T = 71 \quad n = 83 \quad t = 23$$

The formula yields

$$N = \frac{71 \cdot 83}{23} = 256.2$$

and we have a population estimate of about 256 birds.

Further discussion:

1. In the second sample,  $23/83=27.7\%$  of the birds had been tagged.

On the other hand, with our estimate of  $N = 256$  birds and a total of 71 tagged birds around, the ratio percentage of tagged birds is  $71/256=27.7\%$ . These percentage match, which is encouraging.

2. The estimates are very sensitive to the data, because we extrapolate.

If for instance, one more bird without a tag had been captured,  $n$  would change to  $n = 84$ , and the population estimate becomes

$$N = \frac{71 \cdot 84}{23} = 259.3$$

If, on the other hand, if the additional bird had a tag, we now would have  $n = 84$  and  $t = 24$  and the estimate changes to

$$N = \frac{71 \cdot 84}{24} = 248.5$$

This shows we should take population estimates based on this method with a grain of salt.

The mark-recapture method assumes that the animals move around and that we have random samples each time. If the animals all hang out at their respective preferred locations, we do not get a good mix, and the population estimates are tainted. The time between captures needs to be long enough so that we get a good mix, but it shouldn't be too long. Otherwise some tagged animals might die, or other individuals might move into or out of the region of study.

## 8.7 Dilution and Mixtures

We begin with an example: The instructions on a can of frozen pink lemonade concentrate ask to add four parts of water for each part of frozen concentrate. This tells us that no matter how much or how little we want to fix, one fifth of the mixture will be concentrate, and the remainder will be water. In this example, the solute (the frozen concentrate) is available at 100% concentration, while in the mixture it is reduced to  $1/5 = 20\%$  concentration, and we call 5 the dilution factor.

In general mixture problems we follow the amount of the solute (the substance to be dissolved) in each solvent (medium in which the process takes place). A concentration ( $C$ ) is measured as solute ( $s$ ) per volume ( $V$ ), and we have

$$C = \frac{s}{V} \quad \text{or, equivalently} \quad s = CV$$

If we denote the initial values (stock solution) with subscripts 1, and the final product (sample) with subscript 2, we obtain the formula

$$C_1 V_1 = C_2 V_2 (= s) \quad (1)$$

because the solute  $s$  must be the same in both mixtures.

**Example:** We return to the pink lemonade problem, and suppose that we want to prepare two gallons.

We begin with pure concentrate, and thus  $C_1 = 1$ , and  $V_1$  is unknown. As for the final product we know that  $C_2 = 1/5 = 0.2 = 20\%$  (four parts of water per one part of concentrate,  $\frac{1}{1+4}$ ), and we need  $V_2 = 2$  gal. Therefore

$$V_1 = \frac{C_2 V_2}{C_1} = 0.2 \cdot 2 \text{ gal} = 0.4 \text{ gal}$$

Since frozen concentrate usually comes in 12 oz. containers, we should get at least five cans (four is not quite enough, why?).

**Example:** A stock solution contains 25 mg/L of imaginol, we add 10 mL of the stock to 490 mL of water, and we are interested in the concentration of imaginol in the mixture.

Here  $C_1 = 25$  mg/mL,  $V_1 = 10$  mL and  $V_2 = 500$  mL and we obtain

$$\begin{aligned} C_2 &= \frac{C_1 V_1}{V_2} = \frac{25 \text{ mg/mL} \cdot 10 \text{ mL}}{500 \text{ mL}} \\ &= 0.5 \text{ mg/mL} \end{aligned}$$

We now take another look at the notion of a **dilution factor** and we define it as

$$\delta = \frac{C_1}{C_2}$$

It is the stronger concentration divided by the weaker concentration.

But we can obtain the same factor by looking at volumes only. With the use of formula (1) we find that

$$\delta = \frac{C_1}{C_2} = \frac{C_1 V_1}{C_2 V_1} = \frac{C_2 V_2}{C_2 V_1} = \frac{V_2}{V_1}$$

and the dilution factor is the ratio of the larger volume to the smaller volume.

**Example:** We return to the last case.  $V_1 = 10$  mL were dissolved in  $V_2 = 500$  mL. Thus, the dilution factor is

$$\delta = \frac{V_2}{V_1} = \frac{500}{10} = 50$$

and the concentrations behave accordingly:

$$C_2 = \frac{C_1}{\delta} = \frac{25 \text{ mg/mL}}{50} = 0.5 \text{ mg/mL}$$

**Example:** A stock solution contains 18 mg/mL of fictosin and we want to prepare a 30 mL mixture with concentration 3 mg/mL.

We solve this problem by looking at dilution factors. By inspection of the concentrations we see that

$$\delta = \frac{18 \text{ mg/mL}}{3 \text{ mg/mL}} = 6$$

Therefore, using that  $V_2 = 30$  mL, we find that

$$V_1 = \frac{V_2}{\delta} = \frac{30 \text{ mL}}{6} = 5 \text{ mL}$$

Thus, you need to mix 5 mL of the stock solution with 25 mL of water.

### 8.7.1 Serial Dilutions

Sometimes it is necessary to dilute a mixture by a tremendous amount, like putting a droplet into a full bathtub. This is not practical and lab supplies can be very expensive. In order to save cost, such a dilutions are performed in sequence. Dilute the stock solution, mix well, then take a portion of the mixture and dilute it again.

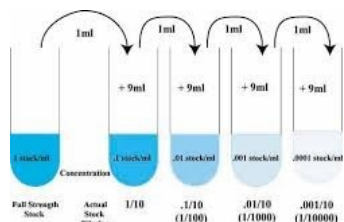


Figure 16: Four 1:10 Serial Dilutions

Each dilution step will have its own dilution factor, and the cumulative dilution factor is the product of the individual factors. For instance, suppose you perform four 1:10 dilutions. The dilution factor is 10 each time, and four such dilutions result in the factor  $\delta = 10^4 = 10,000$  overall.

**Example:** After three 1:200 serial dilutions, a  $10 \mu\text{L}$  droplet contained 53 bacteria. What is the original bacteria concentration?

Here the dilution factor is 200 in each step (one part bacteria mixture, 199 parts broth), and overall the factor becomes

$$\delta = 200^3 = 8,000,000$$

The current concentration is

$$C_2 = \frac{53 \text{ bacteria}}{10 \mu\text{L}} = 5.3 \text{ bacteria}/\mu\text{L}$$

and therefore

$$C_1 = \delta \cdot C_2 = 4.24 \cdot 10^7 \text{ bacteria}/\mu\text{L}$$

which is the equivalent of 42.4 trillion bacteria per liter.

### 8.8 Worked Problems

- In an experiment the Moravian geneticist Gregor Johann Mendel (1822-1884) interbred true yellow round seed peas with true green wrinkled seed peas. The  $F_2$  progeny were distributed as shown in the table

315	yellow round seeds
108	green round seeds
101	yellow wrinkled seeds
32	green wrinkled seeds

Approximate the ratio of phenotypes with whole numbers as  $A : B : C : D$  so that  $A + B + C + D = 16$ .

Solution: The total number of plants is  $315 + 108 + 101 + 32 = 556$ , and we can easily compute the percentages. The sum of the percentages will be one, and if we multiply by 16, we obtain the desired result.

Seeds	Freq.	Perc.	$\times 16$
Yellow round	315	56.7%	9.06
Green rounds	108	19.4%	3.11
Yellow wrinkled	101	18.1%	2.91
Green wrinkled	32	5.8%	0.92
Sum	556	100%	16

The phenotypes follow about a 9:3:3:1 ratio, which is prevalent in genetics.

- The National Forest Headquarters in Roanoke manage the George Washington National Forest (up I-81 North either towards West Virginia or towards the Blue Ridge Parkway) with 1,065,398 acres, and 723,350 acres of the Jefferson National Forest (mostly west of Roanoke, including the Mount Rogers area and the forest land north of Blacksburg, including the Cascades and Mountain Lake). What percentage of the

entire area is the Jefferson National Forest? What is the ratio of the acreages?

Solution: The combined area is 1,788,748 acres (just add) and the percentages become

$$\frac{1,065,398}{1,788,748} = 59.6\%$$

for the George Washington National Forest, and

$$\frac{723,350}{1,788,748} = 40.4\%$$

for the Jefferson National Forest. The ratio is roughly 3:2 because

$$\frac{1,065,398}{723,350} = 1.473 \approx 1.5 = \frac{3}{2}$$

You could also round the percentages to 60% : 40%.

3. A food inspector detected 3.5 mg of arsenic in one kilogram of fish. What is the concentration?

Solution: The ratio arsenic per total mass becomes

$$\frac{3.5 \text{ mg}}{1 \text{ kg}} = \frac{3.5 \text{ mg}}{1,000,000 \text{ mg}} = 3.5 \text{ ppm}$$

The arsenic content in this sample is 3.5 ppm. (Example adapted from Langkamp/Hull).

4. The volume (and not the radius!) of a sphere increases by 50%. How do radius and surface to volume ratio change?

This scenario could come up in the study of cell growth, where ultimately the changes of the surface area to volume ratio are of interest.

Solution: Recall that the volume of a sphere is given by  $V = \frac{4\pi r^3}{3}$  and the surface area is  $S = 4\pi r^2$ . We do not have any specific numbers to work with, which makes the problem more challenging.

We denote the original radius by  $r$ , the original volume by  $V$ , and for the new, bigger sphere we use  $r'$  and  $V'$ , respectively. Then, because  $V' = 1.5V$ , we obtain

$$1.5 = \frac{V'}{V} = \frac{4\pi(r')^3/3}{4\pi/3} = (r')^3$$

Thus,  $r' = \sqrt[3]{1.5} = 1.145$ , and we see that the radius has increased by 14.5%.

At this point we branch off, and offer two solution strategies.

Solution One: Assume that the length units are miraculously chosen so that the original sphere has radius  $r = 1$ . Then the original volume is  $V = \frac{4\pi}{3}$ , the surface area is  $S = 4\pi$ , and the original surface area to volume ratio is  $SVR = \frac{4\pi}{4\pi/3} = 3$ .

The new sphere has radius  $r'$ , and the surface area to volume ratio becomes

$$SVR' = \frac{4\pi(r')^2}{4\pi(r')^2/3} = \frac{3}{r'}$$

Comparing both ratios we get

$$\frac{SVR'}{SVR} = \frac{3/r'}{3} = \frac{1}{r'} = 0.874$$

and we conclude that the surface area to volume ratio has dropped<sup>11</sup> by 12.6%.

---

<sup>11</sup>Recall that  $M = \frac{\text{final value}}{\text{initial value}} = 1 + r$ , and therefore  $r = M - 1 = -0.126$ .



Solution Two: In the geometry chapter we have seen that the surface area to volume ratio for a sphere of an arbitrary radius  $r$  is

$$SVR = \frac{3}{r}$$

and if we compare spheres with respective radii  $r$  and  $r'$ , we find that

$$\frac{SVR'}{SVR} = \frac{3/r'}{3/r} = \frac{r}{r'}$$

But we have already calculated the quantity  $\frac{r'}{r} = 1.5^{1/3}$ , and therefore

$$\frac{SVR'}{SVR} = \frac{r}{r'} = 1.5^{-1/3} = 0.874$$

5. What actual distance appears as 5/8 inches on a 1:24,000 map?

Solution: The distance in reality is

$$24,000 \cdot \frac{5}{8} \text{ in} = 15,000 \text{ in} = 1,250 \text{ ft}$$

6. Connelly's run is a small stream in Radford. On a first capture students caught, marked and released 123 crayfish, on a second capture 9 of 120 fish were marked. Estimate the crayfish population from the data.

Solution: We have  $T = 123$ ,  $n = 120$  and  $t = 9$ , therefore

$$N = \frac{123 \cdot 120}{9} = 1640$$

7. How much lemonade can be made with six 12-oz cans of concentrate, when we still require to use four parts of water of each part of concentrate?

Solution: This one is easy. The total concentrate is

$$6 \text{ cans} \cdot \frac{12 \text{ oz}}{\text{can}} = 72 \text{ oz}$$

Add  $4 \times 72$  oz of water for a total of 360 oz, which is the equivalent of 2.8125 gallons of pink lemonade.

8. You dilute 5 mL of a fantasium stock solution with 75 mL of water. The concentration of the mixture is 6 mg/L, what is the concentration of the stock solution?

Solution: We offer two options.

Solution One: We have  $V_1 = 5$  mL,  $V_2 = 75$  mL and  $C_2 = 6$  mg/L. Therefore

$$\begin{aligned} C_1 &= \frac{C_2 V_2}{V_1} = \frac{6 \cdot 75}{5} \text{ mg/L} \\ &= 150 \text{ mg/L} \end{aligned}$$

Solution Two: The dilution factor is

$$\delta = V_2/V_1 = 75/5 = 15$$

Therefore the stock concentration is

$$C_1 = 15 \cdot 6 \text{ mg/L} = 150 \text{ mg/L}$$

9. You want to prepare a mixture containing 1 g sodium hydroxide (NaOH) from a 0.2 M stock solution. How much stock solution is required. Use that the molar mass of NaOH is 40 g/mol.

Solution: Our mixture must contain  $s = 1$  g of NaOH. Since  $s = CV$ , we get

$$\begin{aligned} V &= \frac{s}{C} = \frac{1 \text{ g}}{0.2 \text{ M}} \\ &= \frac{1 \text{ g}}{0.2 \text{ mol/L}} \cdot \frac{\text{mol}}{40 \text{ g}} = \frac{1}{8} \text{ L} \\ &= 125 \text{ mL} \end{aligned}$$

10. A broth has bacterial density of 200 million bacteria per liter. You need a 20  $\mu$ L droplet with about 5 bacteria. How do you proceed?

Solution: The given bacterial density is  $C_1 = 2 \cdot 10^8$  bacteria/liter, the desired concentration is

$$\begin{aligned} C_2 &= \frac{5 \text{ bact.}}{20 \mu\text{L}} \cdot \frac{1000 \mu\text{L}}{\text{mL}} \cdot \frac{1000 \text{ mL}}{\text{L}} \\ &= 250,000 \text{ bacteria/L} \end{aligned}$$

Therefore the dilution factor becomes

$$\delta = \frac{C_1}{C_2} = \frac{2 \cdot 10^8}{2.5 \cdot 10^5} = 800$$

Serial dilutions appear to be in order. One could, for instance, use two 1:10 dilutions followed by a 1:8 dilution.

## 8.9 Exercises

- Radford University reports that women make up 56% of its student body in the fall of 2014. What is the ratio female:male students?
- It was observed that the sex ratio female:male for the *Alligator mississippiensis* is 5:1. What percentage of alligators will be male?
- You got 20 answers correct on a test of 24 questions. What percentage score did you receive? How many would you have to get correct to get higher than a 90%?
- In a genetic experiment with summer squash 171 plants had a white fruit color, 42 were yellow and 15 were green. Find the white:yellow:green ratio so that the total adds to 16. (white+yellow+green = 16).
- The United States had a population of 304 million in 2008, making up 4.55% of the world population. At the same time Brazil accounted for 2.87% of the world population. What was the population of Brazil?
- The natural abundance of  $^{16}\text{O}$  in the atmosphere is 99.762%, that of  $^{18}\text{O}$  is 0.2% and the natural abundance of  $^{17}\text{O}$  is 0.0377%. What is the ratio  $^{16}\text{O} : ^{17}\text{O} : ^{18}\text{O}$ . Express the abundance of  $^{17}\text{O}$  in ppm (parts per million)?
- The EPA sets the maximum contaminant level of arsenic in drinking water at 10 ppb (parts per billion), and that of fluorides at 4 ppm (parts per million).
  - How many grams of
    - arsenic
    - fluorides
 are allowed in one cubic meter (1000 L) of water?
  - How much uncontaminated water is required to dilute one gram of
    - arsenic
    - fluorides
 safely to EPA standards?
- The U.S. population was 282.16 million in 2000 and 309.35 million in 2010 (both for July 1). The national debt was \$5,674 billion in 2000 and \$13,562 billion in 2010.
  - Find the percentage increase of the population for the decade.
  - Find the percentage increase of the national debt for the decade.
  - Compute the debt per capita for 2000 and for 2010.
  - Compute the percentage change of the per capita debt for the decade.
- You determine that the ambient temperature outside is 274 K. You walk inside your apartment, and the thermometer jumps up to 290 K. Find the percent change of the temperatures.

10. In a test of water quality an automated sample measured 1,020 ppb of phosphorus, a cheaper siphon test registered only 880 ppb. Assuming that the automated test is correct, what is the percentage error of the cheaper test?
11. (a) The volume of a spherical cell increases by 12%; how much does the diameter increase?  
(b) The radius of a spherical cell increases by 2%; how much does the surface to volume ratio change?
12. The blood sugar level of a person increases from 92 mg/dL before lunch to 116 mg/dL after lunch. What is the percentage increase?
13. On a map with scale 1:50,000 two points are 14 cm apart. What is their actual distance?
14. On the Virginia Highway Map one inch on the map is approximately 13 miles. What is the scale of this map.
15. Find the conversion factor when lb/gal is changed to g/mL.
16. The CDC reports that 480,000 Americans die from cigarette smoking each year. Given that the U.S. population is 320 million and that of Virginia is 8.4 million, estimate the annual deaths from smoking in Virginia.
17. On the first day biologists capture 67 fish, mark them and put them back in the pond. The next day they catch 42 fish, 12 of which are marked. Estimate the fish in the pond.
18. You observe birds on an island. On the first day you capture, tag and release 71 birds.  
(a) Three days later you capture 83 birds, 23 of which are tagged. Estimate the bird population on the island.  
(b) Just to be sure, you repeat the experiment on the fourth day. This time you find that 18 of 59 birds are tagged. What is your population estimate?
19. A BIOL 131 class trapped and marked 45 crayfish in Wildwood Park. Three days later, they captured 30. Of these 10 were marked. How many fish would you estimate are in this population?
20. You have a stock solution with concentration 40mg/mL of fantasium. You need 50 mL with 6mg/mL of fantasium. How much stock should be mixed with how much water?
21. It takes three parts of water for each part of frozen orange juice. How much frozen orange juice is required for one gallon of juice?
22. Your stock solution contains 45 mg/mL of fictosin, you need 12mL with concentration 3mg/mL for a patient. How much of the stock should you mix with water?
23. You add 25 mL of a stock solution with concentration 18 mg/mL of fictosin to 225 mL of water. What is the fictosin concentration in the mixture?
24. You have a 5 M stock solution of sucrose. You want to make 100 mL of a 0.4 M solution. How much stock solution should you mix with water?
25. You have 4 mL of a bacterial culture. After performing three 1:80 dilutions you count 192 bacteria in a 10  $\mu$ L sample.

What is the original bacteria concentration measured in bacteria per milliliter? How many bacteria does your 4 mL culture contain?

26. You begin with 5 mL of a bacterial culture. After three 1:50 serial dilutions you find that 1 mL contains 84 bacteria. How many bacteria were in the original sample?

**Answers:**

1.  $14 : 11 \approx 5 : 4$
2. 16.7%
3. 88.3%; 22 (21.6 if partial credit is given)
4.  $12 : 3 : 1$
5. 192 million
6.  $2,625 : 1 : 5.3$  or  $498.8 : 0.19 : 1$  and 377 ppm
7. (a) (i) 10 mg (ii) 4 g  
(b) (i)  $100 \text{ m}^3$  (ii) 250 L
8. (a) 9.6% (b) 139 % (c) \$20, 109.16 and \$43,840.31 (d) 118%
9. 5.8 % (5.8394%)
10. 13.7 % (13.7255%)
11. (a)  $3.85\% \approx 4\%$   
(b) decreases by  $1.96\% \approx 2\%$
12. 26.1%
13. 7 km
14.  $1 : 823,680 \approx 1 : 825,000$
15. 8.345
16. 12,600

17. 234.5
18. (a) 256.2 (b) 232.7
19. 135
20. 7.5 mL stock, 42.5 mL water
21. 1 quart
22. 0.8 mL
23. 1.8 mg/mL
24. 8 mL
25. 9.83 billion per milliliter and 39.3 billion
26. 52.5 million

## 9 Elements of Probability

Processes in nature have a certain degree of randomness and non-predictability. While on for humans the ratio of female to male progeny is about equal, you can find couples with four girls and no boys at all, or families where all children are male. An understanding of such randomness is vital in biology, particularly in the study of genetics. As another application we will look at Hamilton's Rule from evolution.

### 9.1 Probability

For simplicity we assume that our observation (experiment) has only finitely many outcomes, which are all equally likely. Typical examples are rolling a (fair) die, flipping a coin or drawing a card. The sample space  $S$  consists of all possible outcomes, and an event  $E$  is a subset of the sample space.

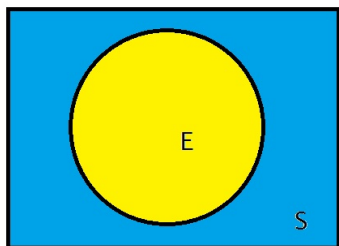


Figure 17: Venn Diagram

Events are often depicted by Venn diagrams. The big blue box depicts the sample space containing all possible outcomes, while the yellow region singles out a specific event.

**Example:** When rolling a die, the outcomes are the numbers one through six, the sample space is the set

$$S = \{1, 2, 3, 4, 5, 6\}$$

and the event of rolling an even number is the set

$$E = \{2, 4, 6\}$$

**Definition:** The probability of an event  $E$  is

$$P(E) = \frac{n(E)}{n(S)}$$

where  $n(S)$  denotes the size of the sample space, that is, the number of possible outcomes, and  $n(E)$  denotes the number of outcomes that make up the event  $E$ . In this definition it is imperative that there are only finitely many and equally likely outcomes.

**Examples:**

1. The probability of rolling a "3" on a fair die is

$$P(E) = \frac{1}{6} = 0.167 ,$$

because  $n(E) = 1$  and  $n(S) = 6$ .

2. A standard deck has 52 cards, two of which are red threes ( $\heartsuit 3$  and  $\diamondsuit 3$ ). We make drawing a red three our event  $E$ . Then the probability of  $E$  is

$$P(E) = \frac{2}{52} = 0.0385 = 3.85\%$$

because  $n(S) = 52$  and  $n(E) = 2$ .

3. In this example we are rolling two dice, a red and a blue one, and we want to find the probability of getting an eight or higher.

Here an outcome is a pair of numbers  $(r, b)$ , with  $r$  being the value on the red die, and  $b$  the value on the blue die. We have 36 possible outcomes as illustrated in Figure 18.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 18: Outcomes for a Pair of Dice

There are 15 possible cases where the outcome is eight or higher; hence the probability of the event is

$$P(E) = \frac{15}{36} = 0.417$$

## 9.2 Laws of Probability

The concept of probability can be extended to spaces with infinitely many outcomes, and to situations where not all outcomes are equally likely.

**Empirical probabilities** are based on collected data. For instance, in the United States the blood types are distributed as follows

	O	A	B	AB
+	37.4%	35.7%	8.5%	3.4%
-	6.6%	6.3%	1.5%	0.6%

Therefore, the probability that a randomly selected person has blood type B+ is  $0.085 = 8.5\%$ .

Rolling dice or drawing cards are examples **theoretical probabilities**. For example, the chances of rolling a "3" are  $1/6$  in theory, but there is no guaranty that when you roll a die 30 times you will get exactly five threes. The *Law of Large Numbers* states that in the long run the empirical probabilities will approach the theoretical probabilities.

For an arbitrary sample space  $S$  and a set of events  $E$ ,  $A$ ,  $B$  we require that the probabilities observe the following three rules:

1.  $P(S) = 1$
2.  $P(E) \geq 0$
3. If  $A$  and  $B$  are exclusive events, that is, if  $A$  and  $B$  cannot occur simultaneously, then<sup>12</sup>

$$P(A \text{ or } B) = P(A) + P(B)$$

As a consequence of these axioms we find that

$$0 \leq P(E) \leq 1$$

holds for any event  $E$ . We say that  $E$  is a *certain event* if  $P(E) = 1$ , and that  $E$  is an *impossible event* if  $P(E) = 0$ .

Moreover, the events

$E$  occurs

$E$  does not occur

are mutually exclusive and they cover the entire sample space (all possibilities). Therefore

$$\begin{aligned} 1 &= P(S) \\ &= P(E \text{ or } (E \text{ does not occur})) \\ &= P(E) + P(E \text{ does not occur}) \end{aligned}$$

and it follows that (negation rule)

$$P(E \text{ does not occur}) = 1 - P(E)$$

**Examples:** We roll a single fair die.

1. The probability of rolling a "7" is zero. It is an impossible event.
2. The probability of rolling a number between one and six is one. It is a certain event, all possible outcomes are covered.

<sup>12</sup>In probability theory one would require that this statement also holds for infinitely many exclusive events.

3. The probability of not getting a "3" is

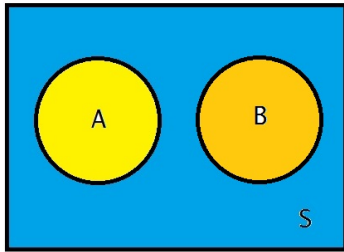
$$\begin{aligned} & P(\text{do not roll a 3}) \\ &= 1 - P(\text{roll a 3}) \\ &= 1 - \frac{1}{6} = \frac{5}{6} = 0.833 \end{aligned}$$

by the negation rule.

The third property of probabilities

$$P(A \text{ or } B) = P(A) + P(B)$$

is known as the additive property, or the "OR-Rule". It requires that the events are mutually exclusive.



In the example of rolling two dice, the probability of rolling a "7" or a "10" is

$$\begin{aligned} & P(\text{roll a 7 or a 10}) \\ &= P(\text{roll a 7}) + P(\text{roll a 10}) \\ &= \frac{6}{36} + \frac{3}{36} = \frac{9}{36} = 0.25 \end{aligned}$$

The values are taken from the table in Figure 18. The addition principle applies, because you cannot roll a "7" and a "10" at the same time.

The story is different when you look at the events

- $A$  roll an "8"
- $B$  the red die shows an even number.

Again, Figure 18 reveals that

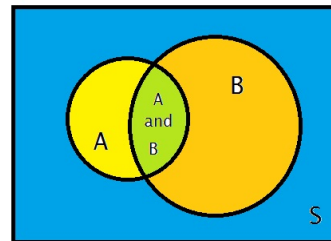
$$\begin{aligned} P(A) &= \frac{5}{36} \\ P(B) &= \frac{18}{36} = \frac{1}{2} \\ P(A \text{ or } B) &= \frac{20}{36} \\ P(A) + P(B) &= \frac{23}{36} \end{aligned}$$

The numbers do not add up,  $\frac{5+18}{36} \neq \frac{20}{36}$ , but this is not in violation of the OR-Rule, because the events are not exclusive!  $A$  and  $B$  occur can at the same time, for instance, when the red die has a "2" and the blue die shows a "6". Three outcomes are doubly counted in the sum  $\frac{5+18}{36} = \frac{23}{36}$ .

### 9.3 Conditional Probability and Independent Events

Very often we want to know the probability of an event  $A$ , given that an event  $B$  has occurred. We express this value as  $P(A|B)$ . In this case  $B$  plays the role of the sample space, and the conditional probability is computed as

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \quad (2)$$



**Example:** We are rolling a red and a blue die again, and let  $A$  be the events of rolling an eight or higher. We have seen before that  $P(A) = \frac{15}{36}$ .

But we peek, and we see that the red die shows a "3". This changes our odds, because we now need a "5" or a "6" on the blue die to make at least eight points. The probability of this is happening is  $\frac{2}{6} = \frac{1}{3}$ . We make  $B$  the event of getting a "3" on the red die, and in light of our new notation, we have

$$\begin{aligned} & P(\text{roll 8 or higher} \mid \text{red die shows "3"}) \\ &= P(A|B) = \frac{1}{3} \end{aligned}$$

We now show how the formula (2) leads to the same result. Inspection of Figure 18 reveals that

$$\begin{aligned} & P(\text{roll 8 or higher and red die is "3"}) \\ &= P(A \text{ and } B) = \frac{2}{36} \quad \text{and} \\ & P(\text{red die shows "3"}) \\ &= P(B) = \frac{1}{6} \end{aligned}$$

and formula (2) results in

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{2/36}{1/6} = \frac{1}{3}$$

If the events  $A$  and  $B$  are exclusive, the two events cannot occur at the same time, which makes " $A$  and  $B$ " an impossible event with probability zero, and

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{0}{P(B)} = 0$$

If the probability of the event  $A$  is not affected by the event  $B$ , we call the events  $A$  and  $B$  *independent*. Then

$$P(A) = P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

and therefore

$$P(A \text{ and } B) = P(A) P(B)$$

This is called the multiplicative rule (or AND-rule) for independent events.

**Example:** When we roll two dice, the result on the blue die is not impacted by the value on the red die, and the outcomes are independent. Therefore the probability of getting an even number on the red die and a five or a six on the blue die equals

$$\begin{aligned} & P((\text{even on red}) \text{ and } (5 \text{ or } 6 \text{ on blue})) \\ &= P(\text{even on red}) \cdot P(5 \text{ or } 6 \text{ on blue}) \\ &= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \end{aligned}$$

This result can be confirmed by looking at Figure 18. There are six outcomes for which the blue die shows a five or a six, and the red die has an even number and the probability is  $\frac{6}{36} = \frac{1}{6}$ , as expected.

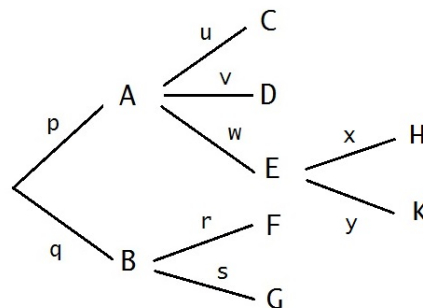


Figure 19: Tree Diagram

Complex situations are often sketched with tree diagrams. In Figure 19 the events are denoted by upper case letters, and the probabilities are shown on the branches. In the figure it is imperative that  $A$  and  $B$  are exclusive, and that  $p + q = 1$  from the OR-rule. Moreover,  $C$ ,  $D$  and  $E$  must be exclusive and  $u + v + w = 1$ , and so on.

As we go down a certain path, we multiply the probabilities. For instance, the probability that  $A$  and  $C$  occur is the product  $p \cdot u$  (AND



rule), and the probability that  $A$ ,  $E$  and  $H$  occur is  $p \cdot w \cdot x$ .

**Example:** A disease is linked to a recessive allele  $a$ . Your uncle (mother's brother) has the illness, but neither do your mother or your maternal grandparents. No one on your father's side of the family carries the disease, and we assume that your father has genotype  $AA$ . What is the probability that the allele, and potentially pass it on to your children?

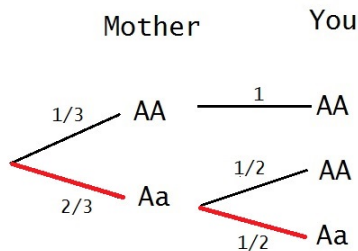
Maternal Grandparents		
	A	a
A	AA	Aa
a	Aa	aa

First of all, since your father has genotype  $AA$ , it is impossible that you have the illness. Secondly, maternal grandparents must both be heterozygotes, and since your mother does not have the disease, the possibility that your mother is  $aa$  can be ruled out.

For the remaining cases, the probabilities for your mother's genotype are  $P(AA) = \frac{1}{3}$  and  $P(Aa) = \frac{2}{3}$ . The probability that you carry the  $a$  allele is  $1/2$  if your mother is  $Aa$ . Therefore the probability that you have genotype  $Aa$  is

$$P = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

The tree diagram will clarify the situation.



## 9.4 Hamilton's Rule

Altruism is a behavior where the an individual (the altruist) makes sacrifices to benefit its kin. Parenting is one such example. The guiding principle is to promote the success of its own genes.

In Hamilton's rule we measure the benefit  $B$  in terms of average additional offspring for the kin, and the cost  $C$  of an altruistic act in terms of the fewer offspring by the altruist. The relatedness factor  $r$  is crucial. Siblings, on average, share 50% of the genes and  $r = 0.5$ . In an aunt/niece relationship, the aunt shares 50% of the genes with the mother, who in turn as 50% of the genes in common with the daughter, which puts the aunt/niece relatedness at  $r = 0.5 \cdot 0.5 = 0.25$  (multiplication principle). For cousins the relatedness factor is  $r = 0.125$ .

$r$	Relationship
0.5	siblings, parent and child
0.25	aunt or uncle with niece or nephew
0.125	cousins

Hamilton's Rule states that altruism is favored if

$$rB > C$$

**Example:** A monkey is under attack by a predator, but it will have a 75% chance of survival if the predator is distracted, else it will get killed. The monkey's brother has a 25% chance of being killed if he distracts the predator. Both would have an average number of four offspring later in life. Should the brother help?

The benefit in this scenario is

$$B = 4 \cdot 0.75 = 3$$

that is, on average three future offspring would be saved (this is an *expected value*). The cost for the brother is

$$C = 4 \cdot 0.25 = 1$$

that is, one future offspring is put at risk. The monkeys are siblings and therefore  $r = 0.5$ . We find that  $rB = 0.5 \cdot 3 = 1.5$ , which is greater than  $C = 1$ , and therefore the brother should distract the predator.

More examples are given in the "Worked Problems" section.

## 9.5 The Birthday Problem

This is a somewhat famous problem, and we include it at this point, although there is no immediate biological connection.

**Problem:** What is the probability that in a group of  $N$  people at least two people have a common birthday? For simplicity, we ignore leap years and assume that the year has 365 days.

In a group of two or three people the odds for a shared birthday are fairly slim, in larger groups the chances become better, and as soon as we have 366 or more people we are guaranteed at least one common birthday<sup>13</sup>.

The problem is easier to solve when we ask for the probability that all birthdays are different, and we make this our event  $E$ . We will now go over the probability of  $E$  for increasing values of  $N$ , the size of the group.

1.  $N = 1$ . With just one person the cannot be a matching birthday, and  $P(E) = 1$
2.  $N = 2$ . If we think of the first person's birthday as a given, then there are 364 different ways for the second person to avoid having the same birthday. Therefore  $P(E) = \frac{364}{365} = 0.9973 = 99.73\%$  and it is very likely that the birthdays are different.

<sup>13</sup>This is an application of the Pigeonhole Principle: We have 365 possible birthdays (pigeonholes) and more than 366 people (pigeons). Therefore, at least two people (pigeons) must share a birthday (pigeonhole).

3.  $N = 3$ . Now a third person enters the rink. The probability of all different birthdays becomes (AND-rule)

$$P(E) = \frac{364}{365} \cdot \frac{363}{365} = 99.18\%$$

4. Adding a fourth person leads to

$$P(E) = \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} = 98.36\%$$

5. If we continue in this fashion, the probability of different birthdays in a group of  $N$  people becomes

$$P(E) = \frac{364 \cdot 363 \cdot 362 \cdots (366 - N)}{365^{N-1}}$$

The table and the graph in Figure 20 summarize the results. The break even point is at  $N = 23$  people. In a group of  $N = 50$  people, the chances of at least one common birthday are at 97%, for a group of 100 people it is almost find to find at least one matching birthday. Test it your favorite group or club, or at a large party!

## 9.6 Worked Problems

1. You flip a coin, what is the probability that you get heads twice.

**Solution:** We denote heads by  $H$  and getting tails by  $T$ . Then the sample space becomes

$$S = \left\{ \begin{array}{cccc} HHH & HHT & HTH & HTT \\ THH & THT & TTH & TTT \end{array} \right\}$$

The sample space has eight possible outcomes, three of which have two heads and one tail. Therefore

$$P(E) = \frac{3}{8} = 0.375 = 37.5\%$$

N	Different Birthdays	Matching Birthdays
5	97.29%	2.71%
10	88.31%	11.69%
15	74.71%	25.29%
20	58.86%	41.14%
23	49.27%	50.73%
25	43.13%	56.87%
30	29.37%	70.63%
40	10.88%	89.12%
50	2.96%	97.04%
75	0.03%	99.97%
100	0.000031%	

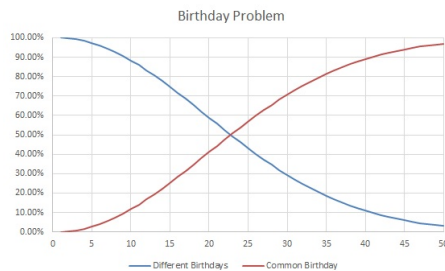
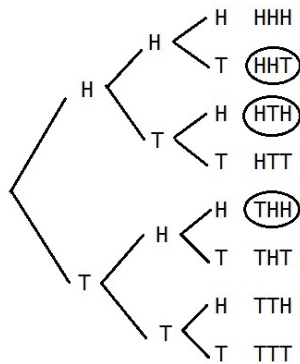


Figure 20: The Birthday Problem

This problem can also be worked with a tree diagram, where all probabilities are  $p = \frac{1}{2}$ .



- Suppose that for one gene we have alleles  $A$  and  $a$ , and that we cross two het-

erozygotes. What is the probability that a progeny has genotype  $aa$ ?

Solution: The possible outcomes can be expressed in a Punnett square, and we see that in one of four cases we obtain genotype  $aa$ . Thus,

$$P(aa) = \frac{1}{4} = 0.25 = 25\%$$

	A	a
A	AA	Aa
a	Aa	aa

- What is the probability of rolling a "7" or a "10" with a pair of dice?

Solution: In reference to Figure 18 we see that  $P("7") = \frac{6}{36}$  and that  $P("10") = \frac{3}{36}$ . The events are exclusive. Therefore

$$\begin{aligned} P("7" \text{ or } "10") \\ &= P("7") + P("10") \\ &= \frac{6+3}{36} = 0.25 \end{aligned}$$

- A family has three children. What is the probability that they have at least one girl.

Solution: The "at least" statement is the stumbling block here.

Question: How can they not have at least one girl?

Answer: If all kids are boys.

We compute the probability that all children are boys first. This can be done by a tree diagram similar to the coin toss problem, or we can argue that the sex of the first, second or third child are all

independent events, and thus the multiplication rule applies. In either case we find

$$P(\text{all boys}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

By the negation rule, the probability of having at least one girl is

$$\begin{aligned} & P(\text{at least girl}) \\ &= P(\text{not all boys}) \\ &= 1 - P(\text{all boys}) = 1 - \frac{1}{8} \\ &= \frac{7}{8} = 87.5\% \end{aligned}$$

5. **Dihybrid Cross.** We are looking at two traits.

Allele  $R$  is dominant over  $r$ ; the phenotype for  $R-$  (this means  $RR$  or  $Rr$ ) is red, and that for  $rr$  is white.

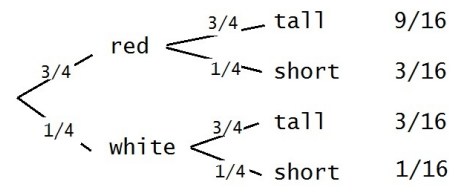
Allele  $T$  is dominant over  $t$ , the phenotype for  $T-$  is tall, and that for  $tt$  is short.

What is the probability of getting short red progeny when we cross  $RrTt \times RrTt$ ?

Solution: The results for each trait are independent and  $P(R-) = \frac{3}{4}$  while  $P(tt) = \frac{1}{4}$ . Therefore

$$\begin{aligned} & P(\text{red and short}) \\ &= P(R-) \cdot P(tt) \\ &= \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16} \\ &= 0.1875 = 18.75\% \end{aligned}$$

As an alternative we can set up a tree diagram.



The diagram confirms the result that short and red progeny has probability 3/16, it also explains why 9:3:3:1 is such an important ratio in genetics.

6. **Population Genetics.** Cystic fibrosis (CF) is linked to a recessive allele. Suppose that one of every 2,500 newborns is affected. What is the probability that a randomly selected person carries the allele without having the disease?

Solution: Our population gene pool has alleles  $A$  and  $a$ . We set  $p = P(A)$  the probability of selecting gene  $A$ , and  $q = P(a)$ .

What is the probability of picking allele  $a$  twice? By the multiplication rule (AND rule) we get

$$P(aa) = q^2,$$

but we also know that

$$P(aa) = \frac{1}{2,500}$$

Thus  $q^2 = \frac{1}{2,500}$  and  $q = \frac{1}{50}$ . The negation rule implies that  $p = P(A) = \frac{49}{50}$ .

The problem calls for  $P(Aa)$ . A person can pick up the  $a$ -allele either as the first or as the second allele. This results in

$$\begin{aligned} P(Aa) &= 2 P(A) P(a) = 2pq \\ &= \frac{2 \cdot 49}{50^2} = 0.0392 \end{aligned}$$

This is essentially a Hardy-Weinberg problem phrased in terms of probabilities, or

you could argue that Hardy-Weinberg is really an application of the rules of probability.

7. **Hamilton's Rule.** A young boy is close to drowning in a heavy surf. Should his sister swim out and try to rescue him if she has a 25% chance of drowning herself? Assume that both would have two children each later in life. (Problem adapted from Campbell.)

Solution: The benefit of the rescue is  $B = 2$ , the two children the boy might have. The cost for the sister is  $C = 2 \cdot 25\% = 0.5$ . They are siblings, therefore  $r = 0.5$  and we get

$$rB = 1 > 0.5 = C$$

and the sister should attempt to rescue her brother. This is based on Hamilton's Rule alone. For real people in an emergency situation there are many other factors to consider.

## 9.7 Exercises

- You flip a coin four times. What is the probability of getting
  - all heads?
  - heads on the first toss, and tails on the other three tosses?
  - heads and tails exactly twice (any order)?
- You roll a pair of dice. What is the probability
  - of rolling a six or higher?
  - that at least one of the dice shows a three?
  - of rolling a pair?
    - of rolling a pair of sixes?
    - that both dice show even numbers?
    - that the sum is an even number?
    - of not getting a ten?
- A family has four children. What is the probability that
  - all of them are girls?
  - at least one is a girl?
  - of having two boys and two girls?
- A city plants 50 oak trees, 75 ash trees and 100 maple trees randomly scattered in a newly designed park. A trail cuts through the park so that 90 trees are to the north of the trail, the remaining trees are to the south. A pair of wrens settles in one of the trees (randomly chooses one of the trees).
  - What is the probability that the birds nest to the north of the trail?
  - What is the probability that the birds nest on a maple tree?
  - What is the probability that the birds nest on a maple tree north of the trail?
  - What is the probability that the birds do not nest in an oak tree?
- A disease is linked to a recessive allele  $a$ .
  - If both parents have genotype  $Aa$ , what is the probability that the child will have the disease?
  - If one parent has the disease, what is the probability that a child will have the disease if the other parent has
    - genotype  $AA$ ?
    - genotype  $Aa$ ?

6. **Dihybrid Cross.** We are looking at two traits. Allele R is dominant over r; the phenotype for R- (RR or Rr) is red, the phenotype for rr is white. Allele T is dominant over t, the phenotype for T- is tall, and for tt it is short. We cross RrTt x RrTt. What is the probability of getting tall white progeny?
7. **Crossing mice.** When yellow mice are bread together, a phenotypic ratio of about two yellow mice for each non-yellow mouse is observed. But when yellow mice are bred with non-yellow mice, the progeny show a 1:1 ratio of yellow to non-yellow mice. Can you explain what's going on here?  
Use Y and y for the alleles, and use that Y is dominant over y, that is, yellow mice have genotype Y—, while non-yellow mice have genotype yy.
8. What is the relatedness factor  $r$  for grandparents and grandchildren in Hamilton's rule?
9. A group of three monkeys - all siblings- is under attack by predators. A fourth monkey, a brother of the three, notices the predators. He has a 80% chance of getting killed if he distracts the predators, but each of the siblings will have a 90% chance of survival if he gives an alarm call. All would have an average number of five offspring in the future. Should the brother help?
10. A young woman needs a kidney transplant due to a rare genetic defect. She has a 90% chance of survival, if she does. Her cousin is a match, but she has a 10% chance of dying if she donates her kidney. Both would have an average of two kids if they live. Should she help her cousin?
- Would the answer change if the woman only had a 80% chance of survival. Use Hamilton's Rule.
11. A bird foraging in a flock with two aunts sees a predator. It has a 20% chance of dying if it gives an alarm call. However, the two aunts have a 75% chance of living if they fly off immediately. Birds in the population have an average of six offspring, but the aunts are already two years old and each has had half of those offspring. Should the young bird give an alarm call?

### Answers

1. (a) 1/16  
(b) 1/16  
(c) 3/8
2. (a) 13/18  
(b) 11/36  
(c) 1/6  
(d) 1/36  
(e) 1/4  
(f) 1/2  
(g) 11/12
3. (a) 1/16  
(b) 15/16  
(c) 3/8
4. (a) 40%  
(b) 44.4%  
(c) 17.8%  
(d) 77.8%
5. (a) 25%  
(b) (i) 0 (ii) 50%

6.  $3/16$
7. The genotype YY is lethal.
8.  $r = 0.25$
9. Yes,  $6,75 > 4$
10. Yes,  $0.225 > 0.2$
11. No,  $1.125 < 1.2$

## 10 The Binomial Distribution

In many cases the observations are "black" or "white", "true" or "false", "either - or" with no in-between. This leads to Bernoulli trials and the binomial distribution. Applications include genetics, medical research and many others.

### 10.1 Bernoulli Trials

We begin with an example. A disease is linked to a recessive allele  $a$ , and both parents have genotype  $Aa$ . In this case the probability that a child will have the disease is 25%. Now suppose that they have three children. What is the probability that none of the children has the disease? What is the probability that one or two have the illness? How likely is it that all have the disease?

Here is another scenario: A basketball player has a 80% free-throw percentage. During a game he steps to the line 12 times. Aside from fatigue, crowd noise, or other factors, what is the probability that he sinks 10 shots?

A **Bernoulli**<sup>14</sup> **trial** is an experiment whose outcome is either success or failure. There is no ethical or moral value attributed to the term "success", it just means that the event in question occurred. The probability of success denoted by  $p$ , that of failure is  $q$ . We then perform a sequence of  $n$  identical such trials, and it is assumed that the outcomes are independent, and that  $p$  and  $q$  remain the same throughout. The question becomes to determine the probability of  $k$  successes

In the recessive disease example, the number of experiments is the number of children, namely  $n = 3$ , and  $p = 0.25$ , if having the disease is the outcome of interest (success). For

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<sup>14</sup>Named for the Swiss mathematician Jacob Bernoulli, 1655-1705.

each new child the cards are shuffled again, and the outcomes are independent.

For the basketball player we have  $n = 12$  and  $p = 0.8$ . It is assumed that the experiments are independent, and that making or missing the basket has no bearing on the result of the next shot.

## 10.2 Binomial Probability Distribution

We now give some background<sup>15</sup> for the formula to compute the probability of  $k$  successes in  $n$  trials when the probability of success is  $p$ .

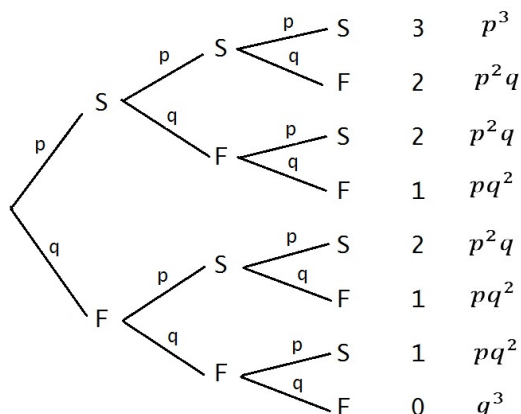


Figure 21: Three Bernoulli Trials

Figure 21 shows the possible results for three Bernoulli trials. Successes are indicated by  $S$ , failures by  $F$ . The column of numbers counts the successes on each branch, and the last column denotes the probability for each case. We see that there is only one way to get three successes, and this happens with probability  $p^3$ . There are three ways to get two successes, each has probability  $p^2q$ . The three ways of having only one success have probability  $pq^2$ , and the one outcome with all failures has probability  $q^3$ .

<sup>15</sup>It is a proof for  $n = 3$  and we do a lot of hand waving for the general case.

This 1 – 3 – 3 – 1 pattern is linked to the computation

$$(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$$

The term  $p^3$  is the probability of three successes, the term  $3p^2q$  represents the probability of two successes, the term  $3pq^2$  is linked to one success, and the last term  $q^3$  is the probability of no success at all.

Similar patterns emerge for other values of  $n$ . If only  $n = 2$  trials are considered, the probabilities are tied to

$$(p + q)^2 = p^2 + 2pq + q^2$$

For  $n = 5$  the tree diagram would get much bigger, but the probabilities would again be related to

$$(p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$

For instance, having three successes and two failures has probability  $10p^3q^2$ , and so on. Notice that  $p + q = 1$ , and considering all cases, the probabilities will always add up to one, because  $(p + q)^n = 1^n = 1$ .

We have argued before that the coefficients in the expansion of  $(p + q)^n$  can be found from Pascal's triangle, and if we want a formula for the probabilities, we need a formula for these numbers.

First we define the **factorial** for positive integers:

$$n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n$$

For instance,

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

$$25! = 1 \cdot 2 \cdot 3 \cdots 24 \cdot 25 = 1.55 \times 10^{25}$$

The convention

$$0! = 1$$



The **binomial coefficient** is defined as

$$\binom{n}{k} = C(n, k) = \frac{n!}{(n-k)! k!}$$

for integers  $0 \leq k \leq n$ . Notice that there is no bar between  $n$  and  $k$  in the parentheses (it is not a fraction). The binomial coefficient is pronounced as “ $n$  choose  $k$ ”, because it stands for the number of ways to select  $k$  items from a set of  $n$  elements<sup>16</sup>, which motivates the “ $C$ ” in the notation  $C(n, k)$ .

**Example:** Let  $n = 5$  and  $k = 3$ , then

$$\binom{5}{3} = \frac{5!}{2!3!} = \frac{120}{2 \cdot 6} = 10$$

There is a more practical way to compute this expression, without the evaluation of factorials:

$$\binom{5}{3} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

If we apply the strategy of the example to any  $C(n, k)$ , we should proceed as follows: First take the smaller of  $k!$  or  $(n - k)!$  and enter the terms in decreasing order in the denominator. Then begin to write  $n!$  in decreasing order in the numerator, and align each number on top with one below, but stop when you run out of matching entries. The value of the resulting fraction is the binomial coefficient. The example below will illustrate this technique some more.

It can be shown that all entries in Pascal's triangle have the form  $\binom{n}{k}$ . Here  $n$  counts

<sup>16</sup>Consider the set  $\{a, b, c, d\}$  of  $n = 4$  elements. Now pick any  $k = 2$  of these. This gives you the options  $\{a, b\}$ ,  $\{a, c\}$ ,  $\{a, d\}$ ,  $\{b, c\}$ ,  $\{b, d\}$  or  $\{c, d\}$ . We have  $6 = C(4, 2)$  ways to do this.

the row, and  $k$  indicates how far you go down a particular row - left to right, or right to left, it doesn't matter because the triangle is symmetric. For both,  $n$  and  $k$ , it is essential to start counting with zero.

n																		
0								1										
1						1		1										
2					1		2		1									
3				1		3		3		1								
4			1		4		6		4		1							
5			1		5		10		10		5		1					
6		1		6		15		20		15		6		1				
7		1		7		21		35		35		21		7		1		
8		1		8		28		56		70		56		28		8		1
	k	0	1	2	3	4	5	6	7	8								

Figure 22: Pascal's Triangle

**Example:** We confirm the highlighted numbers in Figure 22.

The "6" is the third number in the 5th row, but since we start counting at zero, we get  $n = 4$  and  $k = 2$ . It now follows that

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

For the "35" we find that  $n = 7$  and  $k = 3$  and

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

And finally for "28" we obtain  $n = 8$  and  $k = 6$  and

$$\binom{8}{6} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

We are now ready for the big formula. The probability of  $k$  successes in  $n$  Bernoulli trials is

$$b(n, p, k) = \frac{n!}{(n-k)! k!} p^k q^{n-k} \quad (3)$$

where  $p$  is the probability of success, and  $q = 1 - p$  is the probability of failure. This is called the *binomial probability distribution*.

The first fraction in (3) is the binomial coefficient from Pascal's triangle; it counts in how many ways  $k$  successes can be achieved. The last term involves the probabilities. Each success comes with probability  $p$ , and thus  $k$  successes lead to the factor  $p^k$ . If we have  $k$  successes in  $n$  trials, then we also have  $(n - k)$  failures, and these introduce the factor  $q^{n-k}$ . The tree diagram in Figure 21 gives an illustration for  $n = 3$ .

**Example:** Back to the basketball case. What is the probability of 10 successes in 12 trials when the probability of success is 80%?

Here  $n = 12$ ,  $k = 10$  and  $p = 0.8$ . The probability of failure<sup>17</sup> is  $q = 1 - p = 0.2$ , and 10 successes in 12 trials means that we have  $n - k = 2$  failures. For the binomial coefficient we find

$$\binom{12}{10} = \binom{12}{2} = \frac{12 \cdot 11}{2 \cdot 1} = 66$$

The full binomial distribution formula (3) results in

$$\begin{aligned} & b(12, 0.8, 10) \\ &= \frac{12!}{10! 2!} 0.8^{10} 0.2^2 = 66 \cdot 0.1074 \cdot 0.04 \\ &= 0.2835 \end{aligned}$$

### 10.2.1 Calculators and Computers

Graphing calculators can help a great deal with the computations. The following instructions work for TI calculators, and similar steps can be applied on other brands.

The MATH button has a PRB option. Here you find "!" for factorials and "nCr" for the bi-

nomial coefficient. For instance  $C(12, 10)$  can be computed by "12 nCr 10".

The binomial distribution is found under DISTR, and it is called "binompdf". In order to compute  $b(12, 0.8, 10)$ , just enter

$$\text{binompdf}(12, 0.8, 10)$$

and your calculator will produce 0.2835. The command has the structure "binompdf(n,p,k)", and if  $k$  is omitted, "binompdf(n,p)" will display all values for  $k$  from 0 to  $n$ .

EXCEL has a lot of built-in functions. "=FACT(n)" will compute a factorial, the command "=COMBIN(n,k)" will calculate  $C(n, k)$  and "=BINOM.DIST(k,n,p,false)" can be used for binomial distributions.

### 10.2.2 Cumulative Density Distribution

Very often we are interested in more than just the probability for a particular value of  $k$ . For instance, in the basketball problem we might ask for 10 or more baskets in 12 attempts. The cumulative probability density function takes care of this. It adds all the probabilities from  $k = 0$  up to a certain threshold.

The formulas become very complicated, and we illustrate the concept with an example. Let's say we take  $n = 5$  trials and the probability of success is  $p = 0.4$ . In the table below the column labeled "pdf" contains all probabilities for  $k$  from 0 to 5, computed with formula (3) as usual.

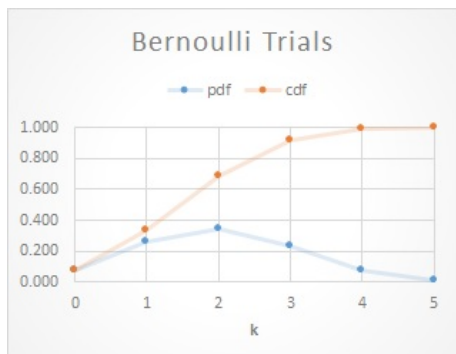
$k$	pdf	cdf
0	0.078	0.078
1	0.259	0.337
2	0.346	0.683
3	0.230	0.913
4	0.077	0.990
5	0.010	1.000

<sup>17</sup>If you make a basket 80% of the time, you miss 20% of the time.

The column "cdf" keeps the running total. For instance, at  $k = 2$  it contains the sum

$$0.078 + 0.259 + 0.346 = 0.683$$

When we reach  $k = 5$ , all cases are covered and the probabilities add to one.



The graph shows both functions. The density function peaks for  $k = 2$ . This is not surprising, because with  $n = 5$  trials and a 40% success rate one would expect  $0.4 \cdot 5 = 2$  successes. The cumulative function adds the pdf values, it increases and reaches 1 when  $k = 5$ .

The command "binomcdf(n,p,k)" will compute cumulative densities on a calculator, and in EXCEL the command is " =BINOM.DIST(k,n,p,true)".

**Example:** A basketball player with a 80% free throw percentage steps to the line 12 times. What is the probability of making 10 or more points?

We need to be careful here. We are interested in 10, 11 or 12 successes, but the cumulative density function will start counting with  $k = 0$ . Hence, and this is a calculator result,

$$\text{binomcdf}(12, 0.8, 9) = 0.442$$

is the probability of making anywhere from none to nine baskets, and therefore the probability of sinking 10 to 12 shots is

$$P(\text{at least 10 baskets})$$

$$\begin{aligned} &= P(\text{not 9 or less baskets}) \\ &= 1 - 0.442 = 0.558 \end{aligned}$$

by the negation rule.

### 10.3 Worked Problems

1. Compute

- (a)  $16!$
- (b)  $\frac{16!}{11!}$
- (c)  $\frac{16!}{5! 11!}$

Solution:

- (a)  $16! = 1 \cdot 2 \cdot 3 \cdots 15 \cdot 16$   
 $= 20,922,789,888,000$
- (b)  $\frac{16!}{11!} = \frac{1 \cdot 2 \cdots 10 \cdot 11 \cdot 12 \cdots 15 \cdot 16}{1 \cdot 2 \cdot 3 \cdots 10 \cdot 11}$   
 $= 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16$   
 $= 524,160$
- (c)  $\frac{16!}{5! 11!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$   
 $= 4,368$

2. Sickle-cell anaemia (SCA) is linked to a recessive allele. Suppose that the parents are both heterozygotes with respect to this trait. What is the probability that two of their three children have the disease?

Solution: Here  $n = 3$  (each child is an experiment!), and we are looking for  $k = 2$  occurrences of the SCA. We also know that the probability of acquiring the disease is  $p = 0.25$ . The binomial distribution formula implies that

$$\begin{aligned} b(3, 0.25, 2) &= \frac{3!}{1! 2!} 0.25^2 0.75 \\ &= 3 \cdot 0.0625 \cdot 0.75 = 0.140625 \\ &\approx 14\% \end{aligned}$$

3. Given that 30% of the students are smokers, what is the probability that a randomly selected group of six students

- (a) consists entirely of non-smokers?
- (b) consists entirely of smokers?
- (c) is evenly split between smokers and non-smokers?
- (d) has at least one smoker?
- (e) has two or more smokers?

Solution: For this problem  $n = 6$  in all cases, and  $p = 0.3$ , if we look for smokers (success). Selecting a non-smoker then has probability  $q = 0.7$ .

- (a) Selecting a non-smoker six times in a row has probability

$$\begin{aligned} P(\text{all non-smokers}) \\ = 0.7^6 = 0.118 \end{aligned}$$

by the multiplication rule for independent events. One can also apply formula (3) with  $k = 0$  (recall that  $0! = 1$  and that  $p^0 = 1$  for any number  $p$ ). We find that

$$\begin{aligned} b(6, 0.3, 0) &= \frac{6!}{6! 0!} 0.3^0 0.7^6 \\ &= 0.7^6 = 0.118 \end{aligned}$$

- (b) Here  $k = 6$  and

$$\begin{aligned} b(6, 0.3, 6) &= \frac{6!}{0! 6!} 0.3^6 0.7^0 \\ &= 0.3^6 = 0.000,729 = 0.07\% \end{aligned}$$

- (c) In this case  $k = 3$  and it follows that

$$\begin{aligned} b(6, 0.3, 3) &= \frac{6!}{3! 3!} 0.3^3 0.7^3 \\ &= 20 \cdot 0.027 \cdot 0.0343 = 0.185 \end{aligned}$$

- (d) If the group has at least one smoker, they cannot be all non-smokers. Using the negation rule and the result from part (a) we obtain

$$\begin{aligned} P(\text{at least one smoker}) \\ &= 1 - P(\text{all non-smokers}) \\ &= 1 - 0.118 = 0.882 \end{aligned}$$

- (e) This problem calls for adding the probabilities for  $k = 2, 3, 4, 5$  and 6. This becomes a little easier, if we negate the statement first and look for the probability of having at most one smoker in the group. We know from part (a) that the probability for  $k = 0$  smokers in the group is 0.118 and for  $k = 1$  we obtain

$$\begin{aligned} b(6, 0.3, 1) &= \frac{6!}{5! 1!} 0.3^1 0.7^5 \\ &= 6 \cdot 0.3 \cdot 0.168 = 0.303 \end{aligned}$$

and having none or just one smoker in the group has probability

$$0.118 + 0.303 = 0.421 = 42.1\%$$

By the way, the same result can be obtained using the cumulative distribution, and the calculator shows<sup>18</sup> that

$$\text{binomcdf}(6, 0.3, 1) = 0.420,175$$

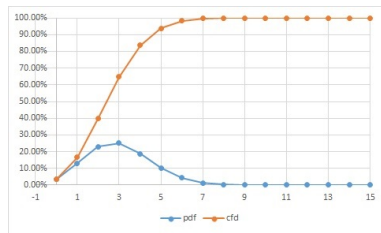
Thus, having two or more smokers in in group occurs with probability (negation principle)

$$1 - 0.420 = 0.580 = 58.0\%$$

---

<sup>18</sup>There is a little rounding discrepancy; 42.0175% is correct.

4. In a 1995 study the CDC reported that 20% of college women have been a victim of rape during their lifetime. Some critics question the validity of this figure, while newer studies seem to support similar values<sup>19</sup>. For the sake of argument, let's stick to the 20% figure for this problem.



The graph shows the probabilities for having  $k$  victims in the group. In a group of 15 one would expect having about three victims in the group<sup>20</sup>, and this has the highest probability. But by chance it could very well happen that we have fewer or more victims in the group. However, having eight or more in a randomly selected group is highly unlikely, as the graph shows.

Here are the questions: What is the probability that in a group of 15 randomly selected college women

- (a) at least one has been a victim of rape?
- (b) exactly one of them has been a victim?
- (c) three or more have been raped?

Solutions:

<sup>19</sup>A CDC study in 2014 reports that 19.3% of women have been victim of rape during their lifetime, about 78% of the rape victims were 25 years old or younger.

<sup>20</sup>20% of 15 is 3, and statisticians call this the expected value.

- (a) "At least one" victim rules out that none has been raped. The probability of not having been victimized is  $q = 1 - p = 0.8$ , and  $0.8^{15} = 0.035$ . Therefore the probability that none has been abused is 3.5%, and the probability that at least one has been raped is (negation rule)

$$1 - 0.035 = 0.965 = 96.5\%$$

- (b) Here  $n = 15$ ,  $p = 0.2$  and  $k = 1$  and we are looking for

$$\begin{aligned} b(15, 0.2, 1) &= \frac{15!}{14! 1!} 0.2^1 0.8^{14} \\ &= 15 \cdot 0.2 \cdot 0.044 \\ &= 0.132 = 13.2\% \end{aligned}$$

- (c) This excludes none, one or two rape victims. The cumulative density distribution yields (calculator and/or the graph above)

$$\text{binomcdf}(15, 0.2, 2) = 0.398$$

that is, having up to two victims in the group has probability 39.8%, and therefore having three or more has probability  $0.602 = 60.2\%$

5. Given that 4% of the population carry the cystic fibrosis allele. What is the probability that in a group of 50 people
- (a) nobody,
  - (b) one person,
  - (c) two people,
  - (d) three or more

carry the CF allele?

Solution: Here  $n = 50$ ,  $p = 0.04$ , and we are looking for the probabilities with

$k = 0$ ,  $k = 1$  and  $k = 2$ . Routine computations show that

$$\begin{aligned}\text{binompdf}(50, 0.04, 0) &= 0.130 \\ \text{binompdf}(50, 0.04, 1) &= 0.271 \\ \text{binompdf}(50, 0.04, 2) &= 0.276\end{aligned}$$

which answers parts (a) - (c). If we add the values, we obtain  $0.677 = 67.7\%$ , which is the probability of having up to two carriers. Hence, the probability of having three or more carriers is

$$1 - 0.677 = 0.323 = 32.3\%$$

6. An experimental medical procedure is 86% successful. What is the probability that it fails for three of twelve patients in a hospital? What is the probability that it fails for three or more patients?

Solution: With an 86% success rate and 12 patients, we expect  $0.86 \cdot 12 = 10.3$  successes, and thus 1.7 failures. But this is not the question. We need the probability of  $k = 9$  successes in  $n = 12$  trials when the probability of success is  $p = 0.86$ , and we find that

$$b(12, 0.86, 9) = 0.155$$

When we ask for three or more failures, we are looking at 9 or less successes. The cumulative density function yields

$$\text{binomcdf}(12, 0.86, 9) = 0.230$$

Our hospital is not doing too well. Its success rate of 75% lags behind the average, and about the bottom quarter of hospitals have similar dismal success rates.

7. A student takes a multiple choice test by random guesses. There are five questions on the test, each with four choices. What is the probability of getting

- (a) four correct answers?
- (b) at least four correct answers?

Follow up: How do the results change when all questions "true or false"?

Solution: Here  $n = 5$ , and for the first questions we have a  $p = 0.25$  chance of success.

- (a)  $k = 4$  and

$$\begin{aligned}b(5, 0.25, 4) &= \frac{5!}{4! 1!} \cdot 0.25^4 \cdot 0.75 \\ &= \frac{5 \cdot 3}{4^5} = \frac{15}{1024} = 0.0147\end{aligned}$$

and the chance of getting four correct answers is about 1.5%.

- (b) This case asks for four or five correct answers. The probability of getting all answers correct is

$$0.25^5 = \frac{1}{1024} = 0.000,98$$

and addition shows that the probability of getting four or more correct answers is

$$\begin{aligned}\frac{15 + 1}{1024} &= \frac{16}{1024} \\ &= \frac{1}{64} = 0.0156 = 1.56\%\end{aligned}$$

In the follow-up the probability of success changes to  $p = 0.5$ , and similar computations show that the probability of getting four problems correct is

$$b(5, 0.5, 4) = 0.15625$$

and the probability of getting four or more correct is

$$\begin{aligned} & b(5, 0.5, 4) + b(5, 0.5, 5) \\ &= 0.15625 + 0.03125 = 0.1875 \end{aligned}$$

## 10.4 Exercises

- Compute
  - $12!$
  - $9!$
  - $\frac{12!}{9!}$
  - $\frac{12!}{9! 3!}$
- You flip a coin four times. What is the probability of getting
  - all tails?
  - tails three times?
  - heads and tails exactly twice (any order)?
- You roll a die five times. What is the probability of rolling sixes twice?
- A family has five children. What is the probability that they have four girls and one boy?
- Suppose that Democrats and Republicans are evenly split on campus. What is the probability that 12 randomly selected students are split evenly into six supporters of each party?
- A multiple choice test has 5 questions with four choices each. What is the probability that you get
  - four correct answers,
  - at least four correct answers

with random guesses?

- An experimental medical procedure is 90% successful. What is the probability that it fails on three of twelve patients in a hospital?
- Given that 4% of the population carry the recessive cystic fibrosis allele. What is the probability that in a group of 25 people
  - nobody
  - one person
  - two people
 carry the allele?

## Answers

- 479,001,600
  - 362,880
  - 1,320
  - 220
- $1/16$
  - $1/4$
  - $3/8$
- 16.1%
- 15.6%
- 22.6%
- (a) 1.465% , (b) 1.5625%
- 8.5%
- (a) 36.0%, (b) 37.5%,  
(c) 18.8%

## 11 Functions

Functions are a very important concept in mathematics, and they are usually introduced in Pre-Calculus. In the next chapters we will look at linear, exponential, logarithmic and power functions which play an important part in biological modeling. The purpose of this section is to introduce the function concept. We touched loosely on this topic in the context of graphing already.

**Definition:** A function  $f$  is a rule which assigns to each admissible input  $x$  exactly one output  $y$ . The *domain* of a function is the set of all admissible inputs, and the *range* consists of all possible output values. We write<sup>21</sup>

$$y = f(x)$$

pronounced as "f of x", where  $y$  is the output assigned to the input  $x$ .



The black box concept is often used to visualize functions. An input  $x$  is submitted to the black box, and the rule will assign an output  $y$ . The notation  $f(x)$  emphasizes that the result is the output of the function  $f$  from the input  $x$ .

It is customary say that the input is the *independent variable*, and the output is the *dependent variable*, as it depends on the input.

**Example (Square Roots):** Here the rule ( $f$ ) is taking square roots. The input  $x = 4$  results

<sup>21</sup>  $f$  is the function,  $f(x)$  is the formula for the output. There is a fine difference here, and many a math major struggles with that too.

in the output  $y = 2$ , the input  $x = 25$  generates the output  $y = 5$ , and so on.

For any input  $x$ , the output  $y$  from the function will be  $\sqrt{x}$ . In function notation we write

$$f(x) = \sqrt{x}$$

and the initial examples become

$$f(4) = 2 \quad \text{and} \quad f(25) = 5$$

We cannot take square roots of negative numbers. Thus, the domain of this function is the interval  $0 \leq x < \infty$ , and since taking square roots will never produce a negative result, the range is the interval  $0 \leq y < \infty$  as well.

**Example (An Exponential Function):** Assign the output  $y = 2^x$  to every input  $x$ . For instance, the input  $x = 3$  results in the output  $y = 2^3 = 8$ , for  $x = -2$  we get  $y = 2^{-2} = \frac{1}{4} = 0.25$  and  $x = 0$  produces the output  $y = 2^0 = 1$ .

In function notation we have

$$f(x) = 2^x$$

and our initial examples can be expressed as

$$f(3) = 8 \quad f(-2) = \frac{1}{4} \quad f(0) = 1$$

The domain of this function consists of all real numbers<sup>22</sup>, and the range are the positive numbers ( $y > 0$ ).

**Example: A Function Defined by a Table.** In this example we define a function by

<sup>22</sup> In the Algebra chapter we have looked at  $a^x$  for positive integers  $x$ , and then we extended the meaning of  $a^x$  to negative integers  $x$ , and later on to fractional exponents. It requires some serious Calculus to define  $a^x$  when  $x$  cannot be written as a fraction.



listing all inputs and their associated outputs.

Input	Output
$x$	$f(x)$
1	4
2	6
4	9
5	4
8	5

This means that  $x = 1$  is mapped to  $f(1) = 4$ ,  $x = 2$  is mapped to  $f(2) = 6$  and so on. The requirement that each admissible input has exactly one output is met. The domain of this function is the set of numbers  $D = \{1, 2, 4, 5, 8\}$ . The function is not defined for any other values. The range of the function are the numbers  $R = \{4, 5, 6, 9\}$ , and the fact that  $x = 1$  and  $x = 5$  have the same output is not in violation the definition of a function.

**Several Variables.** There is really no reason to restrict the input of a function to single numbers. For example, the binomial distribution formula

$$b(n, p, k) = \frac{n!}{(n-k)! k!} p^k q^{n-k}$$

can be viewed as a function of three variables. The input consists of the triple  $(n, p, k)$ , where  $n$  is a positive integer,  $0 \leq p \leq 1$  is a real number, and  $0 \leq k \leq n$  is an integer (this is a description of the domain). The output is the value computed from the formula on the right, and it can be shown that the range of this function is the interval  $0 \leq y \leq 1$ .

**General Functions.** The function concept is not restricted to numbers only. For example, you can construct a function by assigning the date of birth (output) to each person (input). When you classify organisms by species, you assign to each organism (input) its scientific name (output). Sometimes it is

useful to interpret those kinds of examples in terms of functions, sometimes using functions makes simple things unnecessarily abstract and complicated. In the sequel we will restrict ourselves to functions tied to numbers.

## 11.1 Graphs of Functions

From your experience with graphing calculators you know that the graph of a function should be some kind of a curve. This is true in many cases, because most of the time the output can be described by a formula. The functions

$$\begin{aligned} f(x) &= \sqrt{x} \\ f(x) &= \frac{4-x}{2} \\ f(x) &= 2^x \end{aligned}$$

serve as examples. In this case you can enter the formula, and the graphing calculator<sup>23</sup> will produce a nice curve.

In these graphs the inputs lie on the  $x$ -axis and the outputs are located on the  $y$ -axis. In order to illustrate the input-output relation, select an input  $x$  on the  $x$ -axis, move up or down to the curve, and then over to the  $y$  axis. This number is the output  $y$ . Figure 23 illustrates this notion for the function  $f(x) = \sqrt{x}$  with input  $x = 6.25$  and output  $y = \sqrt{6.25} = 2.5$ .

You can visualize the domain of a function by projecting the graph of the function down onto the  $x$ -axis, and its range by projection onto the  $y$ -axis. For example, the function given in Figure 24 has domain  $1 \leq x \leq 4$  and range  $1 \leq y \leq 5$ .

You may have noticed than in any of those graphs you never have a situation where one

<sup>23</sup>Before the advent of graphing calculators people would compute  $f(x)$  for selected values of  $x$  (T-table), plot the points, and then connect the dots by a smooth curve.

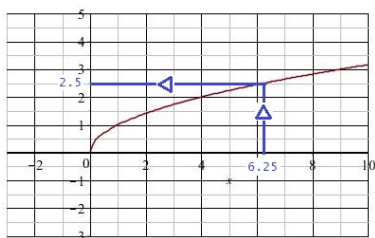


Figure 23:  $f(x) = \sqrt{x}$

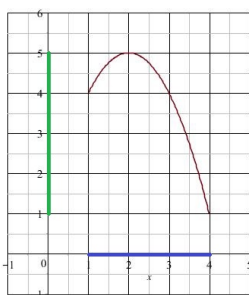


Figure 24: Domain/Range

and the same  $x$  value comes with two different points on the graph. This does not occur, because one input cannot have two different outputs. This idea is called the *Vertical Line Test*<sup>24</sup>. Moreover, any curve which passes the Vertical Line Test can be used to define a function graphically. Just apply the principle of Figure 23.

If a function is given in form of a table, a scatter plot is the most appropriate way to display its graph.

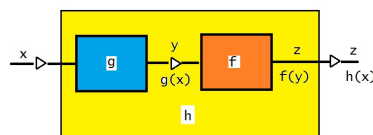
## 11.2 Composition and Inverse Functions

This section is included because exponential functions and logarithmic functions, which will be covered later on, are inverses of each other.

<sup>24</sup>Any vertical line intersects with the graph at most once.

The composition of functions is needed to explain inverse functions. Pre-Calculus usually covers many aspects of functions in great detail; here we limit ourselves to the composition and to inverse functions.

**Composition.** In the composition of functions, the output of one function is used as an input for another function, as illustrated in figure below.



The input  $x$  is fed into the function  $g$  and it produces the output  $y = g(x)$ . This value is then used as input for the function  $f$ . The output becomes

$$z = f(y) = f(g(x))$$

Overall, the input  $x$  results in the output  $z = f(g(x))$ . By combining the two functions we have created a new input-output relationship, that is, we have formed a new function  $h$ . This function is called the composition of  $f$  and  $g$ , and it is commonly denoted as  $h = f \circ g$ . In particular,

$$h(x) = (f \circ g)(x) = f(g(x))$$

**Example:** Take  $f(x) = \sqrt{x}$  and  $g(x) = 2x - 3$ .

The function  $f$  takes square roots of the input, therefore if the input is  $g(x)$ , the output becomes

$$f(g(x)) = \sqrt{g(x)}$$

But we have a formula for  $g(x)$ , and once we substitute we obtain

$$h(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{2x - 3}$$

and the composition is  $h(x) = \sqrt{2x - 3}$ .

**Example:** If  $f(x) = 1/x$  and  $g(x) = 1 - x^2$ , then

$$f(g(x)) = \frac{1}{g(x)} = \frac{1}{1 - x^2}$$

and

$$g(f(x)) = 1 - f(x)^2 = 1 - \frac{1}{x^2}$$

Clearly, the results are different, and  $f \circ g \neq g \circ f$  is the rule, rather than the exception.

**Example:** The temperature  $T$  in degree Celsius  $x$  meters above the ground is given by

$$T(x) = 20 - 0.01x$$

i.e. at the ground level the temperature is  $25^\circ\text{C}$ , and it decreases by one degree Celsius for every elevation gain of 100 m. The height in meters of a projectile  $t$  seconds after its launch is given by

$$h(t) = 80t - 4.9t^2$$

In order to find the air temperature at the location of the projectile as a function of time, we have to first determine the height and then compute the temperature. This results in

$$\begin{aligned} T(h(t)) &= T(80t - 4.9t^2) \\ &= 20 - 0.01(80t - 4.9t^2) \\ &= 20 - 0.8t + 0.049t^2 \end{aligned}$$

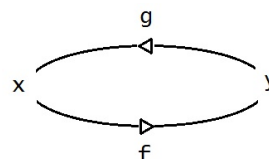
This is a composition of functions. The output of the height function is the input to the temperature function.

**Inverse Functions.** When we construct the inverse of a function, we aim to find a function which "undoes" the effects of the original function. For instance, if  $f$  takes the input 5 to 8, then the inverse function should take 8 back to 5.

**Definition:** A function  $g$  is called the inverse function of  $f$  if

$$g(f(x)) = x \quad \text{and} \quad f(g(y)) = y \quad (4)$$

is valid for all  $x$  in the domain of  $f$  and for all  $y$  in the range of  $f$ .



A few remarks are in order:

1. The notation  $f^{-1}$  is frequently used to denote the inverse of a function. But this can be misleading, because

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

that is, the inverse of a function is not its reciprocal. For this reason we denote an inverse by  $g$ .

2. Suppose that  $f$  takes the number  $a$  to  $b$ , then the inverse will take  $b$  back to  $a$ . In function notation we have

$$f(a) = b \quad \text{and} \quad g(b) = a$$

If we combine these statements, we get

$$a = g(b) = g(f(a))$$

and we see that the composition of inverse functions results in the identity function.

3. Most of the time, if  $g(f(x)) = x$  works out, the other requirements for inverse functions will fall into place. But in some cases it is important to pay attention to domain and range of  $f$  and  $g$  in order to avoid pitfalls.

4. Not every function has an inverse. In the example of tabular data we had  $f(1) = 4$  and  $f(5) = 4$ . It is impossible to construct an inverse function, because we cannot map the value 4 back to 1 and to 5 at the same time (a function  $g$  can only have one output).

In mathematics we call functions for which different inputs always result in different outputs *one-to-one*, and this is the class of functions which have an inverse. We do not want to get into further detail; usually we will encounter some "red flags" as we attempt to compute an inverse.

**Example:** The functions

$$f(x) = 2x - 5 \quad \text{and} \quad g(x) = \frac{x + 5}{2}$$

are inverses of each other. The relationship

$$\begin{aligned} g(f(x)) &= g(2x - 5) \\ &= \frac{(2x - 5) + 5}{2} \\ &= \frac{2x}{2} = x \end{aligned}$$

holds for all  $x$ , and  $f(g(y)) = y$  works in a similar fashion, and there are no issues with domains and ranges.

**Example:** The functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = \sqrt{x}$$

are inverses, provided that, and this is important, we exclude negative numbers. Verification:

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2} = |x| = x$$

The equation  $\sqrt{x^2} = |x|$  holds for all numbers, positive or negative, but the last step requires

that  $x \geq 0$ . Result: For positive numbers the square root of a square is the original number!

For the converse we have

$$f(g(y)) = f(\sqrt{y}) = (\sqrt{y})^2 = y$$

No problem here.  $y \geq 0$  is required for the domain of  $g$ .

Notice that the parabola  $f(x) = x^2$  is not one-to-one. For instance,  $f(4) = 16$  and  $f(-4) = 16$ , and we can only go back from  $y = 16$  to  $x = 4$ , if we exclude the negative  $x = -4$ .

**Example:**  $f(x) = \frac{1}{x}$  is its own inverse:

$$f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = x$$

The reciprocal of the reciprocal is the original number.

Usually, inverse functions are nicely paired on a calculator and connected with the "2ND" key.  $x^2$  comes with  $\sqrt{\phantom{x}}$ , LOG with  $10^x$ , LN with  $e^x$ , and the same goes with the trigonometric functions. If you experiment with your calculator, you will observe that in most cases no matter what messy number the original function produces, the inverse will take you back to where you started:

$$\begin{aligned} \log 412.7 &= 2.615634469 \\ 10^{\text{Ans}} &= 412.7 \end{aligned}$$

The **construction** of an inverse requires to recover  $x$  from  $y = f(x)$ . In other words, we have to solve the equation  $y = f(x)$  for  $x$ .

**Example:** Let  $f(x) = \frac{2x}{x-1}$ . to find the inverse we have to solve

$$y = \frac{2x}{x-1}$$

It's a long process, but it can be done (the steps should be evident):

$$\begin{aligned} y &= \frac{2x}{x-1} \\ (x-1)y &= 2x \\ xy - y &= 2x \\ xy - 2x &= y \\ (y-2)x &= y \\ x &= \frac{y}{y-2} \end{aligned}$$

and the inverse is  $x = g(y) = \frac{y}{y-2}$ , or by renaming the variable, it becomes

$$g(x) = \frac{x}{x-2}$$

Technically, still we have to verify that  $g$  meets all the requirements of an inverse function. This includes to confirm that

$$g(f(x)) = \frac{\frac{2x}{x-1}}{\frac{2x}{x-1} - 2} = x$$

Secondly,  $g$  is not defined for  $x = 2$ , the reason being that there is not a single value for  $x$  which makes  $f(x) = 2$ . Conversely,  $f$  is undefined for  $x = 1$ , and  $g(x) \neq 1$  for any selection of  $x$ .

### 11.3 Worked Problems

1. A function is defined by the formula

$$f(x) = x + \frac{1}{x}$$

- (a) Describe the the function verbally.
- (b) What are the respective outputs for the inputs  $x = 1$ ,  $x = 2$ ,  $x = -2$ ,  $x = 4$  and  $x = \frac{1}{4}$ ?
- (c) Graph the function for  $-4 \leq x \leq 4$ .
- (d) What is the domain of the function?

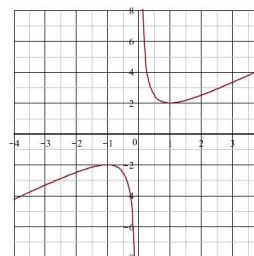
- (e) Does the function have an inverse?

Solutions:

- (a) The function takes the sum of the input and its reciprocal.
- (b)

$$\begin{aligned} f(1) &= 1 + 1 = 2 \\ f(2) &= 2 + \frac{1}{2} = 2.5 \\ f(-2) &= -2 + \frac{1}{-2} = -2.5 \\ f(4) &= 4 + \frac{1}{4} = 4.25 \\ f\left(\frac{1}{4}\right) &= \frac{1}{4} + \frac{1}{\frac{1}{4}} = 4.25 \end{aligned}$$

- (c) The graph is



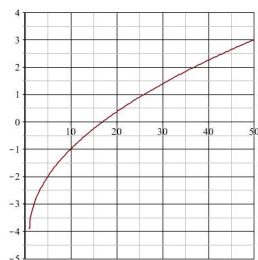
- (d)  $x = 0$  has to be excluded from the domain, as its reciprocal is undefined. All other real numbers are permitted.
- (e) The function cannot have an inverse. For instance,  $x = 4$  and  $x = \frac{1}{4}$  have the same output  $y = 4.25$ , and it is impossible to go back from 4.25. In math terms: The inverse does not exist, because the function is not one-to-one.

2. A function is defined as

$$f(x) = \sqrt{x-1} - 4$$

- (a) Sketch the graph of this function.
- (b) What are the domain and the range of the function?
- (c) Construct the inverse of the function.

Solutions:



- (a)
- (b)  $x \geq 1$  is required, so that the square root exits. Therefore, the domain is  $\{x \geq 1\}$ .  $y = -4$  is the smallest possible value (it occurs for  $x = 1$ ). Therefore, the range is  $\{y \geq -4\}$ .
- (c) We begin with  $y = \sqrt{x-1} - 4$  and solve the equation for  $x$ :

$$\begin{aligned} y &= \sqrt{x-1} - 4 \\ \sqrt{x-1} &= y + 4 \\ x - 1 &= (y + 4)^2 \\ x &= 1 + (y + 4)^2 \end{aligned}$$

Thus, after we switch  $x$  and  $y$ , the inverse function becomes

$$g(x) = 1 + (x + 4)^2$$

The condition  $y \geq -4$  is implied in the first step. Therefore the domain of  $g$  is the set  $\{x \geq -4\}$ .

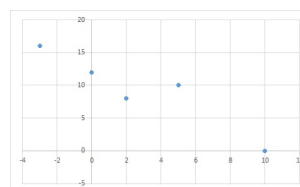
3. A function is defined by the data below.

Input	Output
$x$	$f(x)$
-3	16
0	12
2	8
5	10
10	0

- (a) Compute  $f(0)$ ,  $f(2)$  and  $f(f(5))$ .
- (b) Sketch the graph of the function.

Solutions:

- (a) Directly from the table we obtain that  $f(0) = 12$  and  $f(2) = 8$ . Since  $f(5) = 10$ , it follows that  $f(f(5)) = f(10) = 0$ .
- (b) A scatter plot is most appropriate for tabular data.



4. Let  $f(x) = x(1-2x)$ ,  $g(x) = \frac{1}{2}(1-x)$  and  $h(x) = 1-2x$ . Determine

- (a)  $f(g(x))$
- (b)  $g(h(x))$
- (c)  $h(h(x))$

Solutions:

- (a) First we formally substitute  $g(x)$  for  $x$  in the definition of  $f$ , then we apply the definition of  $g$ . Simplification will do the rest.

$$f(g(x)) = g(x)(1 - 2g(x))$$

$$\begin{aligned}
&= \frac{1}{2}(1-x)(1-2\frac{1}{2}(1-x)) \\
&= \frac{1}{2}(1-x)(1-(1-x)) \\
&= \frac{1}{2}x(1-x)
\end{aligned}$$

(b)

$$\begin{aligned}
g(h(x)) &= \frac{1}{2}(1-h(x)) \\
&= \frac{1}{2}(1-(1-2x)) \\
&= \frac{1}{2}2x = x
\end{aligned}$$

which shows that  $g$  and  $h$  are inverses.

(c) Sometimes it is beneficial to apply a function twice. Here we obtain

$$\begin{aligned}
h(h(x)) &= 1-2h(x) \\
&= 1-2(1-2x) \\
&= 1-2+4x \\
&= 4x-1
\end{aligned}$$

## 11.4 Exercises

1. Let  $f(x) = x^2 - 4$ .

- Describe the the function verbally.
- What are the respective outputs for the inputs  $x = 1$ ,  $x = 2$ ,  $x = -2$ ,  $x = 4$ ?
- Graph the function for  $-5 \leq x \leq 5$ .
- What is the domain of the function?
- Does the function have an inverse?

2. Let  $f(x) = \frac{x}{10} + 4$

- Describe the function verbally.
- Compute  $f(0)$ ,  $f(20)$ ,  $f(-10)$ .
- Sketch the graph of  $f$ .

(d) Find  $x$  such that  $f(x) = 0$

(e) Construct the inverse for  $f$ .

3. Let  $f(x) = \frac{x}{x+1}$

- Compute  $f(0)$ ,  $f(1)$ ,  $f(99)$  and  $f(-3)$
- Sketch the graph of  $f$ .
- Construct the inverse for  $f$ .

4. A function is defined by the data below.

Input $x$	Output $f(x)$
-2	3
0	1
3	6
6	6
10	5

- Compute  $f(0)$ ,  $f(10)$  and  $f(f(f(6)))$ .
- Sketch the graph of the function.

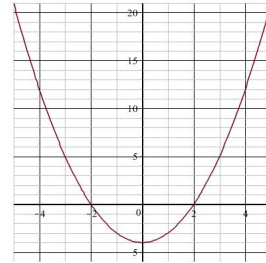
5. Let  $f(x) = \sqrt{x}$ , and let  $g(x) = x^2 + 1$ .  
Form

- $f(g(x))$  and
- $g(f(x))$

6. Confirm that the functions  $f(x) = \frac{1}{x-1}$  and  $g(x) = 1 + \frac{1}{x}$  are inverses of each other.

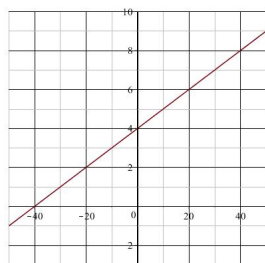
## Answers

- The function subtracts 4 from the square of the input.
  - 3, 0, 0, 12, respectively.

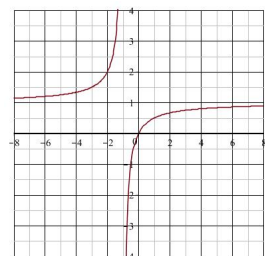


(c)

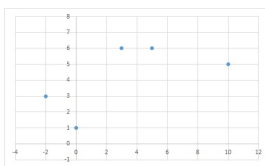
- (d) All real numbers.  
 (e) No.
2. (a) The function takes one tenth of the input and then adds 4 to it.  
 (b) 4, 6 and 3, respectively.



- (c)  
 (d)  $x = -40$   
 (e)  $g(x) = 10x - 40$
3. (a)  $0, \frac{1}{2} = 0.5, \frac{99}{100} = 0.99$  and  $\frac{3}{2} = 1.5$ , respectively.



- (b)  
 (c)  $g(x) = \frac{x}{1-x} = -\frac{x}{x-1}$
4. (a) 1, 5 and 6, respectively.



- (b)
5. (a)  $f(g(x)) = \sqrt{x^2 + 1}$   
 (b)  $g(f(x)) = x + 1$
6. Form compositions to obtain  $f(g(x)) = x$  and  $g(f(x)) = x$ .

## 12 Linear Functions

Linear models are the most natural approach in many problem solving situations.

For instance, if the cost for a wedding increases by \$200 for eight additional guests, then twelve more guests will raise the cost by \$300. It's natural, it's logical and we haven't even said anything about linear relationships.

Linear models are often used as a first step in complex situations, and biology is no exception. We will touch upon examples from climate change, population growth, temperature regulation and anatomy.

### 12.1 Lines

Most everybody is familiar with

$$y = mx + b$$

as an equation for a line. Here

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{\text{rise}}{\text{run}} \end{aligned}$$

is the *slope* of the line. It is also called the *gradient*, although in mathematics this term is usually reserved for multi-variate functions. In most applications the slope represents a *rate*.

What distinguishes a (straight) line from all other curves, is that the slope remains the same throughout. It does not matter which points  $P$  and  $Q$  are selected on the line, the ratio of the differences of the  $y$ -coordinates ( $\Delta y$ ) to the differences of the  $x$ -coordinates ( $\Delta x$ ) always remains the same.

The term  $b$  is the  $y$ -intercept. It can be found by setting  $x = 0$ . In a graph it is value where the line crosses the  $y$ -axis.

**Example (Temperature Conversion):** The relation between the Celsius (C) and Fahren-



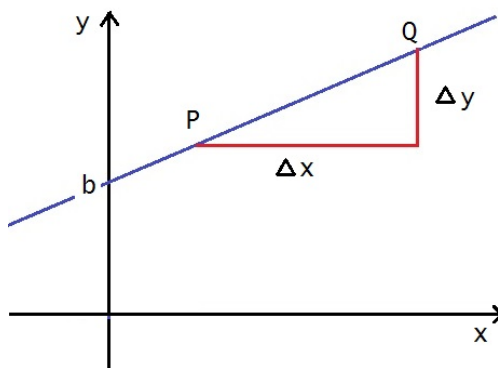


Figure 25: Line

heit (F) temperature is given by the equation

$$F = 1.8C + 32$$

When we set  $y = F$  and  $x = C$  we have an equation of a line

$$y = 1.8x + 32$$

The slope is  $m = 1.8$ ; it tells us that for each  $1^\circ$  increment on the Celsius scale, the Fahrenheit temperature will go up by  $1.8^\circ$ .

In order to illustrate the  $\Delta$  notation, we have to pick two points. So, for instance  $x = 10^\circ C$  is the equivalent of  $y = 50^\circ F$  and  $x = 35^\circ C$  corresponds to  $y = 95^\circ F$ . For these two data we get

$$\begin{aligned}\Delta C &= \Delta x = 35 - 10 = 25 \\ \Delta F &= \Delta y = 95 - 50 = 45\end{aligned}$$

that is, a 25 degree increase in the Celsius scale comes with a 45 degree raise of the Fahrenheit temperature, and the ratio becomes

$$m = \frac{\Delta y}{\Delta x} = \frac{45}{25} = 1.8,$$

as expected.

The  $y$ -intercept is  $b = 32$ . Water freezes at  $x = 0^\circ C$ , which is the equivalent of  $32^\circ F$ .

**Example (Cost of a Wedding):** If  $x$  is the number of guests and  $y$  is the cost of a wedding, the statement that eight more guests increase the cost by \$200 translates into  $\Delta y = 200$  when  $\Delta x = 8$ . The slope becomes

$$m = \frac{\Delta y}{\Delta x} = \frac{200}{8} = 25$$

This is a rate, it measures the cost per person.

The relationship  $m = \frac{\Delta y}{\Delta x}$  can be rephrased as

$$\Delta y = m \Delta x$$

and with  $m = 25$ , we find that  $\Delta y = 25\Delta x$ , which tells us that if the number of attendees changes by  $\Delta x$ , the cost will change by  $\Delta y = 25 \Delta x$ .

We have no information about the fixed cost, i.e. expenses which are independent of the number of guests. Thus,  $b$  is unknown and we cannot set up an equation of the cost of a wedding in terms of the number of guests.

**Linear Functions.** When we express a line in function form,

$$f(x) = mx + b$$

we call it a *linear function*. Examples include

$$\begin{aligned}f(x) &= 4x - 9 \\ f(x) &= 6 - 0.5x \\ f(x) &= \frac{4 - x}{2}\end{aligned}$$

There is really no significant difference between  $f(x) = 2x - 3$  and  $y = 2x - 3$ . The first notation emphasizes the function concept, the second is more appropriate for graphs in the  $xy$ -plane. If you want to find the value of a linear function for a specific point  $x$ , the function notation comes in handy. For example, let

$$f(x) = 2x - 3$$

then the value for  $x = 2$  is

$$f(2) = 2 \cdot 2 - 3 = 1$$

The alternative would be writing

$$y = 2x - 3$$

and then

$$y|_{x=2} = 2 \cdot 2 - 3 = 1$$

This is all very technical, and hybrids, such as  $y(2) = 1$  are acceptable.

**Tables.** Consider the data in the table

$x$	$y$
-2	14
0	20
2	26
4	32
6	38

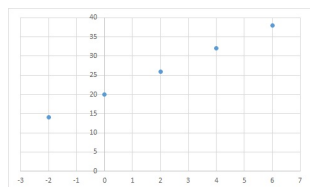
We notice that the data on the left increase by 2, and those on the right increase in steps of 6. This is a clear indication that we are dealing with a line! On the left we have  $\Delta x = 2$  and on the right we get  $\Delta y = 6$ . Therefore,

$$m = \frac{\Delta y}{\Delta x} = \frac{6}{2} = 3$$

When  $x = 0$  we see that  $y = 20 = b$ , which is the  $y$ -intercept, and the equation for the data in the table is

$$y = 3x + 20$$

A graph confirms the linear structure of the data.



## 12.2 Point-Slope Form

A line is completely determined if we know two points which belong to the line. It is also determined if we know just one point, as well as the slope.

Let's begin with the latter. Suppose that the slope of a line is  $m$ , and that the point  $P(x_0, y_0)$  belongs to the line. Then the equation of this line is

$$y - y_0 = m(x - x_0) \quad (5)$$

This equation is known as the *point-slope form* of a line, as it requires the knowledge of a point and the slope.

**Example:** A line has slope  $m = 3$ , and it contains the point  $P(2, 1)$ . Its equation is

$$y - 1 = 3(x - 2)$$

which can be rewritten as

$$y = 3x - 5$$

Let's confirm the result: Obviously, the slope is  $m = 3$ , and when we set  $x = 2$ , we find that  $y = 3 \cdot 2 - 5 = 1$ , as desired.

There is an alternative solution. We know that  $m = 3$ , therefore the equation becomes

$$y = 3x + b$$

When we set  $x = 2$  and  $y = 1$ , the substitution yields

$$1 = 3 \cdot 2 + b$$

and solving for  $b$  results in  $b = -5$ . Hence, the equation is  $y = 3x - 5$ , as before.

**Two Points.** Two points also determine a line, but the corresponding formula becomes

more involved<sup>25</sup>. Usually it is easiest to use the two points to determine the slope, and then apply the point-slope formula (5) using any of the two points.

**Example:** Let  $P(1, 4)$  and  $Q(5, 2)$ . Then the slope of the line connecting  $P$  and  $Q$  is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - 4}{5 - 1} = -\frac{1}{2}$$

Once  $m$  is found, we can apply formula (5) and obtain that

$$y - 4 = -\frac{1}{2}(x - 1)$$

In slope-intercept form the line becomes

$$y = -\frac{1}{2}x + \frac{9}{2} = -0.5x + 4.5$$

If we use the point  $Q$  in the point slope formula instead, we obtain

$$y - 2 = -\frac{1}{2}(x - 5)$$

which is equivalent  $y = -0.5x + 4.5$ . This illustrates that we can use either point the point-slope formula.

We test our result: If  $x = 1$ , then  $y = -0.5 + 4.5 = 4$  and for  $x = 5$  we find that  $y = -0.5 \cdot 5 + 4.5 = -2.5 + 4.5 = 2$ , as expected.

### Derivation of the Point-Slope Formula.

Suppose that  $P(x_0, y_0)$  is a point on the line, and that  $Q(x, y)$  is any other point on this line.

<sup>25</sup>The line containing the points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  can be defined by

$$\frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2}$$

or by

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

When we compute the slope between  $P$  and  $Q$ , we find that

$$m = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}$$

Multiplication yields formula (5)

$$y - y_0 = m(x - x_0)$$

## 12.3 Almost Linear Data

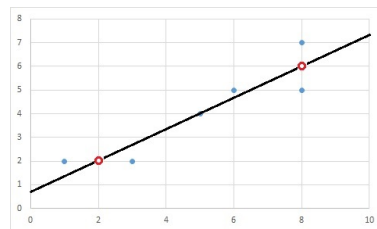
Data in real life are not perfect. There is variation in nature and there are measurement errors. Sometimes you have a lot of data, and there appears to be a linear relationship, but the numbers don't fit perfectly. The goal then becomes to find an approximating line. This topic is carefully studied in regression analysis. This workbook is not the place for a detailed analysis. Instead, we shall use an intuitive approach, the straight-edge method.

The steps are simple:

1. Graph the data points carefully.
2. Trust your intuition, and draw a line which best fits the data.
3. Identify two points on the line - near opposite ends for the sake of stability.
4. Use the points from Step 3 to write an equation for the approximating line.

**Example:** The data and with a graph are given below.

$x$	1	3	5	6	8	8
$y$	2	2	4	5	5	7



The estimated interpolating line contains the points  $P(2, 2)$  and  $Q(8, 6)$ . The slope of the interpolating line is

$$m = \frac{6 - 2}{8 - 2} = \frac{4}{6} = \frac{2}{3}$$

and the line becomes

$$y - 2 = \frac{2}{3}(x - 2)$$

which can be rewritten as

$$y = \frac{2}{3}x + \frac{2}{3} \quad (6)$$

We see that the line contains one data point exactly, otherwise it overshoots or undershoots the data. The table below lists data, predictions and errors.

$x$	$y$	predicted	error
1	2	1.33	-0.67
3	2	2.67	0.67
5	4	4	0.00
6	5	4.67	-0.33
8	5	6	1.00
8	7	6	-1.00

The predictions are obtained by substituting the  $x$ -values into the formula (6). These are the corresponding points on the line. The errors (residuals) subtract the actual data from the predictions. For instance, the observation for  $x = 1$  is  $y = 2$ , the prediction by formula is  $y = 1.33$ , the residual is  $1.33 - 2 = -0.67$ .

Our approach is ambiguous. Different people may come up different lines, and many pretty good answers are possible. Regression analysis identifies the line which minimizes the squares of the errors. This technique is well understood and built into many of computational devices.

On a TI calculator go to STAT and Edit to enter the data into the lists  $L_1$  and  $L_2$ . Then

go to STAT and CALC, and select the Lin-Reg(ax+b) option.

In the above example, the calculator will produce  $a = 0.6395$  and  $b = 0.8627$ . The data can be displayed with the STAT PLOT options (turn the plots on), and the line can be included using the usual "Y=" key for graphing.

In EXCEL you should list the data in columns and produce a scatter plot. If you right-click on a data point, the "Add Trendline" option will show up. Select "Linear", and if you opt for "Display Equation on chart", an equation of the line will show up on the graph.

Calculator and EXCEL will produce identical results, because both are the least squares solution, and if you compare the computer answer

$$y = 0.6395x + 0.8627$$

to our estimation (6)

$$y = 0.6667x + 0.6667$$

you will see that both lines are fairly close, and that our estimation was pretty good, particularly at the upper end.

$x$	$y$	least squares	our estimate
1	2	1.50	1.33
3	2	2.78	2.67
5	4	4.06	4.00
6	5	4.70	4.67
8	5	5.98	6
8	7	5.98	6

## 12.4 Worked Problems

1. The global mean temperature has risen from  $14.00^\circ\text{C}$  in the 1970s to  $14.51^\circ\text{C}$  in the 2000s. Estimate the mean temperature for the 2020s.

Solution: The temperature has increased by  $0.51^\circ\text{C}$  in three decades, which translates into a  $0.17^\circ\text{C}$  increase per decade.

Two decades down the road we expect a rise of another  $0.34^{\circ}C$ , which brings the total average up to  $14.85^{\circ}C$ .

The problem is solved, and it shows that using linear models comes naturally. Without making it explicit, we have used the techniques of lines and linear functions: Given are two temperature data, three decades apart. This results in the slope

$$\begin{aligned} m &= \frac{\Delta \text{ temp}}{\Delta \text{ time}} \\ &= \frac{14.51^{\circ}C - 14.00^{\circ}C}{3 \text{ decades}} \\ &= 0.17^{\circ}C/\text{decade} \end{aligned}$$

If we denote temperature  $x$  decades past the 2000s by  $T(x)$ , our points become  $(-3, 14.00)$  and  $(0, 14.51)$ , and since we have determined  $m$  already, we can apply the point-slope form of a line.

$$T(x) - 14.51 = 0.17x$$

Therefore

$$T(x) = 0.17x + 14.51$$

and two decades past the 2000s the temperature will be  $T(2) = 14.51 + 0.34 = 14.85$ .

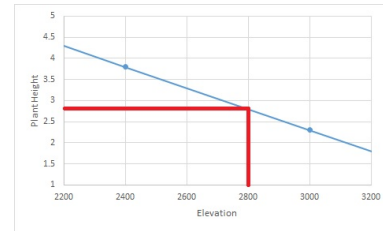
2. At an elevation of 2,400m above sea level a salix drammondiana plant (Drummond's willow) grew 3.8m tall, while at 3000m the height of the same species was only 2.3m. Estimate the height at 2,800m above sea level.

Solution: We offer three solution strategies, which all lead to the same (correct) answer.

Solution One: The height of the plant drops by  $3.8 - 2.3 = 1.5$  meters when

the elevation is raised by 600m, which is equivalent to a loss of 0.25m for every 100m increase in elevation. Changing from 2,400m to 2,800m raises the elevation by 400m, it thus decreases the height of the plant by 1m, and we expect the willow to be 2.8m tall.

Solution Two: A graph says it all. The given information can be translated into the points  $(2400, 3.8)$  and  $(3000, 2.3)$ . Connect the points by a line, and identify the height for 2,800m.



Solution Three: We denote the height of the plant by  $y$  and the elevation above sea level by  $x$ . Then our data are  $P(2400, 3.8)$  and  $Q(3000, 2.3)$ . The slope of the line is

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} = \frac{2.3 - 3.8}{3000 - 2400} = \frac{-1.5}{600} \\ &= -0.0025 \end{aligned}$$

The point-slope formula using  $m$  and the point  $P$  yields

$$y - 3.8 = -0.0025(x - 2,400)$$

which is equivalent to (no reason to distribute the  $-0.0025$ )

$$y = 3.8 - 0.0025(x - 2,400)$$

When  $x = 2,800$  we obtain

$$\begin{aligned} y &= 3.8 - 0.0025(2,800 - 2,400) \\ &= 3.8 - 1 = 2.8 \end{aligned}$$

3. A population of 3,000 fish increases by 160 fish annually. What is the population three years from now? When will it reach 4,000 fish?

Solution: We denote the population by  $p$ , and the time measured in years from by  $t$ . The current ( $t = 0$ ) population is 3,000, which makes this number the  $p$ -intercept. The annual increase sets the slope at  $m = 160$  fish per year. Therefore,

$$p = 160t + 3,000$$

In three years ( $t = 3$ ) we have

$$p = 160 \cdot 3 + 3,000 = 3,480$$

and our population will be 3,480 fish.

To answer the second question we set  $p = 4000$  and solve for  $t$ :

$$\begin{aligned} 4,000 &= 160t + 3,000 \\ 160t &= 1000 \\ t &= \frac{1,000}{160} = 6.25 \end{aligned}$$

and it will take about years and three month until the population reaches 4,000 fish.

4. Find an equation of the line form the data in the table.

(a)

$x$	$y$
-1	6
0	10
1	14
2	18
3	22

(b)

$x$	$y$
-1	3.1
0	2.8
1	2.5
2	2.2
3	1.9

(c)

$x$	$y$
25	17
30	21
35	25
40	29
45	33

Solution:

- (a) The data for  $x$  increase by one ( $\Delta x = 1$ ), and in this case  $m = \frac{\Delta y}{\Delta x} = \Delta y$ , and the  $y$ -increments are automatically the slope. Thus we have  $m = 4$ . The  $y$  intercept is found from  $x = 0$ , and we see that  $b = 10$ . Hence the equation becomes

$$y = 4x + 10$$

- (b) Again,  $\Delta x = 1$ , and since the  $y$ -values decrease by 0.3 each time, we have  $m = \Delta y = -0.3$ . The  $y$ -intercept is located at 2.8, which results in the equation

$$y = -0.3x + 2.8$$

- (c) Here  $\Delta x = 5$  and  $\Delta y = 4$ . Thus  $m = \frac{\Delta y}{\Delta x} = \frac{4}{5} = 0.8$ .

We could backtrack in the table and find  $y$ -values for  $x = 20$ ,  $x = 15$  and so on until we reach  $x = 0$  and then identify the  $y$ -intercept that way. As an alternative we use the point-slope

formula with  $m = 0.8$  and  $P(30, 21)$  and obtain

$$y - 21 = 0.8(x - 30)$$

which can be rewritten as

$$y = 0.8x - 3$$

In any of these exercises we can check the results by comparison of the original data to the formula. For instance, in part (c) substitute any of the  $x$ -values, say  $x = 40$ , into the formula and calculate  $y$ . Here we obtain

$$y = 0.8 \cdot 40 - 3 = 32 - 3 = 29$$

which agrees with the value in the table.

5. Complete the table.

(a)

$x$	$y$
0	0.2
2	0.8
4	
6	
8	

(b)

$x$	$y$
1980	
1990	412
2000	
2010	348
2020	

(c)

$x$	$y$
500	
600	0.86
700	
800	
900	0.74

Solution: It is understood that we use a linear model in each case. Secondly, the  $x$ -increments are constant in all tables, and we just need to focus on the numbers on the right.

- (a) Here  $y$  increases by 0.6 for the first values. If we continue this pattern we obtain

$x$	$y$
0	0.2
2	0.8
4	1.4
6	2.0
8	2.6

Follow-up: It can be shown that  $y = 0.3x + 0.2$

- (b) Here we see that in two steps  $y$  decreases by 64. Thus in each step  $y$  should decrease by 32, which is carried out below.

$x$	$y$
1980	444
1990	412
2000	380
2010	348
2020	316

Follow-up:  $y = 380 - 3.2(x - 2000)$ , if you are curious about a formula.

- (c) Let's add variety and solve this problem by a different method. We are given the points  $P(600, 0.86)$  and  $Q(900, 0.74)$ . Therefore the slope is

$$\begin{aligned} m &= \frac{0.74 - 0.86}{900 - 600} = \frac{-0.12}{300} \\ &= -\frac{1}{2,500} = -0.000,4 \end{aligned}$$

The point-slope formula yields

$$y - 0.86 = -\frac{x - 600}{2,500}$$

and therefore

$$y = 0.86 - \frac{x - 600}{2,500}$$

Now we can plug in values for  $x$  and determine  $y$ . For instance  $x = 800$  leads to

$$\begin{aligned} y &= 0.86 - \frac{800 - 600}{2,500} \\ &= 0.86 - \frac{200}{2,500} \\ &= 0.86 - 0.08 = 0.78 \end{aligned}$$

Proceeding similarly for the other values we obtain

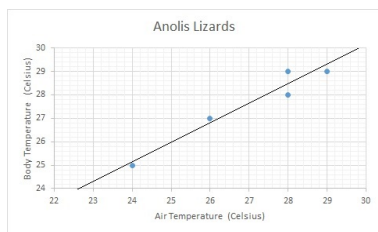
$x$	$y$
500	0.90
600	0.86
700	0.82
800	0.78
900	0.74

6. The body temperatures of Anolis lizards have been recorded in a shady environment at different temperatures with the results given in the table below. Graph the data, determine an approximating line and estimate its slope. What is the biological significance of the slope?

Air Temp	24	26	28	28	29
Body Temp	25	27	28	29	29

Both temperatures are measured in degree Celsius.

Solution:

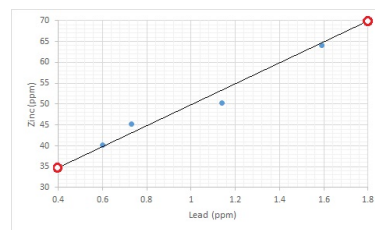


The slope of the approximating line is  $m = \frac{6}{7.2} = \frac{5}{6} = 0.83$ , and we conclude that for each one degree increase of the air temperature, the body temperature of the lizards will increase by about  $0.83^\circ\text{C}$ . Note: The least squares line found by calculator is  $y = 0.81x + 5.66$ .

7. A study of heavy metal concentrations in fish had the results listed below (Spokane River, Langkamp/Hull, some data omitted). Find an approximating line.

Lead (ppm)	0.73	1.14	0.60	1.59
Zinc (pp)	45.3	50.3	40.2	64.0

Solution:



Using the points  $P(0.4, 35)$  and  $Q(1.8, 70)$  we find that

$$m = \frac{\Delta y}{\Delta x} = \frac{70 - 35}{1.8 - 0.4} = \frac{35}{1.4} = 25$$

Thus,  $y - 35 = 25(x - 0.4)$ , which leads to

$$y = 25x + 25$$

Note: The least squares regression line (calculator) is  $y = 22.5x + 27.1$ .

One would expect that, depending on the size of the fish, lead and zinc concentrations would remain proportional. If that were the case, the estimating line would have to pass through the origin, which is not the case in our example. More data should clarify the situation.



## 12.5 Exercises

1. Assume that  $y$  changes linearly with  $x$ . Complete the table below and find an equation of the line.

$x$	$y$
16	40
20	
24	
28	34
32	

2. Find an equation for the line with  $x$ -intercept at  $x = 12$  and  $y$ -intercept at  $y = 4$ .
3. Find an equation of the line containing the points  $P(4, 17)$  and  $Q(10, 5)$ .
4. Write an equation for the line with slope 0.3 and containing the point  $P(120, 45)$ . What is the  $y$ -intercept of this line?
5. A plant is 72 cm tall and it grows at 3 cm per week. How long will it take until it is 84 cm tall?
6. At 10:00 a.m. the water level is 2.5 feet below flood stage, and it rises at 1.5 inches per hour. At what time will the water level reach flood stage?
7. The farming of aquatic organisms, totaled 45.7 million metric tonnes (Mmt) worldwide in 2000. The amount is increasing by 2.7 Mmt each year. Use a linear model to predict when the yield will reach 100 Mmt.
8. Use the straightedge method to approximate the data in the graph by a linear function.

$x$	0	1	3	4	4
$y$	78	80	86	90	88

## Answers

1.  $y = -\frac{1}{2}x + 48$

$x$	$y$
16	40
20	38
24	36
28	34
32	32

2.  $y = 4 - \frac{1}{3}x$

3.  $y = -2x + 25$

4.  $y = 0.3x + 9$ ,  $b = 9$

5. 4 weeks

6. 6 a.m. the next day

7. about 2020

8.  $y = 3x + 77$ ,  
answers may vary

## 13 The Common Logarithm

Logarithms play a very important role in mathematics, mainly in the context of solving exponential equations. We already came across logarithmic scales in the context of graphing, and in the application section we will look at the pH value from chemistry and at earth quake scales. We will focus on the common logarithm, which is the logarithm with base 10.

### 13.1 Introduction

Suppose that  $x > 0$  is a positive number. Our goal is to find a number  $y$  such that

$$x = 10^y$$

This number  $y$  is the logarithm of  $x$ , and we will write  $y = \log x$ .

For some numbers  $x$  this is a no-brainer:

#### Examples:

1. Let  $x = 100$ . Since  $100 = 10^2$ , we see that  $y = 2$ , and we have  $\log 100 = 2$ .
2. Let  $x = 100,000,000,000$ . We see that  $x = 10^{11}$ , and therefore  $\log 100,000,000,000 = 11$ .
3. Take  $x = 0.001$ . Then  $x = 10^{-3}$  and  $\log 0.001 = -3$ .

For integer powers of 10, the logarithm counts by how many places we have moved the decimal point starting with 1.0 - positive if we move it to the right, negative otherwise.

$x$	0.001	0.01	0.1	1	10	100	1000
$\log x$	-3	-2	-1	0	1	2	3

Whenever the number  $x$  is a power of 10, integer or not, it is straightforward to determine the logarithm. For instance, take

$$x = \sqrt{10} = 10^{1/2}$$

and therefore,  $\log \sqrt{10} = \frac{1}{2}$ , or in decimals we have  $\log 3.162 = 0.5$ .

The challenge is working with numbers that are not easily identified as powers of 10. For instance, if  $x = 75$ , it takes some work to come up with the result<sup>26</sup>

$$75 = 10^{1.875}$$

and therefore  $\log 75 = 1.875$ .

### 13.2 Definition

From the introduction it should be clear that logarithms are closely linked to exponents, and the basic laws for exponents

$$a^n a^m = a^{n+m} \quad (7)$$

$$(a^n)^m = a^{nm} \quad (8)$$

will be frequently used in this chapter.

Here is a compact form of the definition of the logarithm:

**Definition:** Let  $x > 0$ , then

$$y = \log x \quad \Leftrightarrow \quad x = 10^y \quad (9)$$

where  $\Leftrightarrow$  means "if and only if"; both statements are equivalent.

The equivalence in the definition allows to transform statements from logarithmic form to exponential form and vice versa. For example,

$$12 = 10^x$$

is a statement in exponential form, its logarithmic equivalent is

$$x = \log 12$$

If we convert the statement that  $10^0 = 1$  to logarithmic form, we find that  $\log 1 = 0$ ,

<sup>26</sup>Here and in other places we will round to about three decimal places, and all numerical statements will have small rounding errors.

and conversion of  $10^1 = 10$  leads to  $\log 10 = 1$ . In summary, we have

$$\log 1 = 0 \quad \log 10 = 1$$

which are two very important logarithms.

### 13.2.1 Estimation of Logarithms

The observation that

$$2^{10} = 1,024 \approx 1,000 = 10^3$$

allows us to estimate  $\log 2$  conveniently. We take the power  $\frac{1}{10}$  on both sides and use the second basic law (8):

$$2 = (2^{10})^{1/10} \approx (10^3)^{1/10} = 10^{3/10}$$

Therefore, converting to logarithmic form, we conclude that

$$\log 2 \approx \frac{3}{10} = 0.3$$

The estimate that  $2 \approx 10^{0.3}$  has numerous corollaries:

1.  $4 = 2^2 \approx (10^{0.3})^2 = 10^{0.6}$  and thus  $\log 4 \approx 0.6$ .
2.  $8 = 2^3 \approx (10^{0.3})^3 = 10^{0.9}$  and therefore  $\log 8 \approx 0.9$ .
3. Division yields

$$5 = \frac{10}{2} \approx \frac{10}{10^{0.3}} = 10^{0.7}$$

and we find that  $\log 5 \approx 0.7$ .

4.  $\frac{1}{2} = 2^{-1} \approx (10^{0.3})^{-1} = 10^{-0.3}$  and thus  $\log \frac{1}{2} \approx -0.3$ , or in decimals we have  $\log 0.5 \approx -0.3$ .
5.  $0.2 = \frac{1}{5} = 5^{-1} \approx (10^{0.7})^{-1} = 10^{-0.7}$  and  $\log 0.2 \approx -0.7$ .

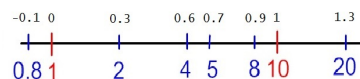


Figure 26: Logarithmic Scale

6.  $20 = 2 \cdot 10 \approx 10^{0.3} 10^1 = 10^{1.3}$ , which implies that  $\log 20 \approx 1.3$

The logarithm is a build-in function on most calculators, and comparison to calculator results shows that our estimates are fairly good.

$$\log 2 = 0.301,030 \approx 0.3$$

$$\log 4 = 0.602,060 \approx 0.6$$

$$\log 5 = 0.698,970 \approx 0.7$$

$$\log 8 = 0.903,090 \approx 0.9$$

A graph of the logarithm function is shown in Figure 27. Notice, that it is defined for positive  $x$  only, and that logarithms are negative when  $0 < x < 1$ .

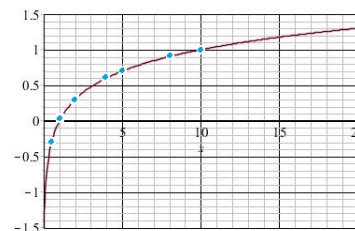


Figure 27: Graph of  $y = \log x$

## 13.3 Properties

In working the examples and exercises you may already have noticed some properties of the logarithm.

**Identities.** The definition of the logarithm is an equivalence, and we can take one side and substitute it into the other. By doing so we obtain

$$x = 10^y = 10^{\log x}$$

as well as

$$y = \log x = \log 10^y$$

After renaming the variables we obtain the two identities

$$a = 10^{\log a} \quad (10)$$

$$a = \log 10^a \quad (11)$$

The first identity requires that  $a > 0$ , the second works for any number  $a$ .

**Examples:**

$$\begin{aligned} 7.5 &= 10^{\log 7.5} = 10^{0.875} \\ -0.6 &= \log 10^{-0.6} = \log 0.251 \end{aligned}$$

**Basic Rules.** Logarithms, like exponential expressions, follow a number of rules. The two rules which parallel the basic rules of exponentials are

$$\log(a \cdot b) = \log a + \log b \quad (12)$$

$$\log a^n = n \log a \quad (13)$$

**Examples:**

$$\begin{aligned} \log 15 &= \log(3 \cdot 5) = \log 3 + \log 5 \\ &= 0.477 + 0.699 = 1.176 \end{aligned}$$

$$\begin{aligned} \log 800 &= \log(8 \cdot 100) = \log 8 + \log 100 \\ &\approx 0.9 + 2 = 2.9 \end{aligned}$$

$$\begin{aligned} \log 64 &= \log 2^6 = 6 \log 2 \\ &\approx 6 \cdot 0.3 = 1.8 \end{aligned}$$

$$\begin{aligned} \log \sqrt[3]{4} &= \log 4^{1/3} = \frac{1}{3} \log 4 \\ &\approx \frac{1}{3} 0.6 = 0.2 \end{aligned}$$

**Verification of the Basic Rules for Logarithms.** For positive numbers  $a$  and  $b$  we set

$$x = \log a \quad \text{and} \quad y = \log b$$

Then, by switching over to exponential form, we get the equivalent relationships

$$a = 10^x \quad \text{and} \quad b = 10^y$$

Now we multiply and apply (7) to obtain

$$ab = 10^x \cdot 10^y = 10^{x+y}$$

which in logarithmic form becomes

$$x + y = \log(a \cdot b)$$

Thus we have shown that

$$\log(a \cdot b) = x + y = \log a + \log b$$

which is the first basic law for logarithms (12).

In order to confirm (13), we form

$$a^n = (10^x)^n = 10^{nx}$$

Then

$$\log a^n = nx = n \log a$$

**Other Identities.** Two more identities are useful with logarithms

$$\begin{aligned} \log \frac{1}{a} &= -\log a \\ \log \frac{a}{b} &= \log a - \log b \end{aligned}$$

The first rule works because  $\frac{1}{a} = a^{-1}$ , and (13) implies that

$$\log \frac{1}{a} = \log a^{-1} = -\log a$$

For the second rule we note that  $\frac{a}{b} = a \cdot \frac{1}{b}$ , and thus

$$\log \frac{a}{b} = \log a + \log \frac{1}{b} = \log a - \log b$$

**Example:** We can estimate  $\log 0.25$  by either rule.

The first rule, along with  $\log 4 \approx 0.6$  implies that

$$\begin{aligned}\log 0.25 &= \log \frac{1}{4} = -\log 4 \\ &\approx -0.6\end{aligned}$$

while the second rule, along with

$$\log 25 = \log 5^2 = 2 \log 5 \approx 2 \cdot 0.7 = 1.4$$

shows that

$$\begin{aligned}\log 0.25 &= \log \frac{25}{100} = \log 25 - \log 100 \\ &\approx 1.4 - 2 = -0.6\end{aligned}$$

### 13.3.1 Scientific Notation

This section is not intended as an efficient method to represent a number in scientific notation. It rather shows a connection between the logarithm and scientific notation.

Recall that in scientific notation we express a number  $x > 0$  in the form

$$x = a \cdot 10^n$$

where  $1 \leq a < 10$  is the mantissa, and where the exponent  $n$  is an integer. Using our new rules (12) and (13), we find that

$$\log x = \log a + \log 10^n = n + \log a$$

If  $n$  is positive, matters are straightforward. The integer part of the logarithm is the exponent  $n$ , and the decimals can be used to compute the mantissa. If  $n$  is negative, we have to subtract an extra "1".

**Example:** Take  $x = 5,636$ . Then

$$\log 5,636 = 3.751$$

This tells us that  $n = 3$  and  $\log a = 0.751$ , which implies that  $a = 10^{0.751} = 5.636$ , and in scientific notation we find that

$$x = a 10^n = 5.636 \cdot 10^3$$

**Example:** Let  $x = 0.000,145$ . Then

$$\log x = -3.839 = -4 + 0.161$$

Thus,  $n = -4$  and  $\log a = 0.161$ . We find that  $a = 10^{0.161} = 1.45$ , and  $x = 1.45 \cdot 10^{-4}$ .

Here it was necessary to subtract an extra "1", because the tail  $-0.839$  does not lie between zero and one. Actually,

$$10^{-0.839} = 0.145$$

and we have to multiply by 10 in order to obtain the required range for  $a$ .

**Example:** If  $\log x = 8.45$ , we can tell without a calculator that  $x$  is on the order of  $10^8$ , that is,  $x$  equals a few hundred million. On the other hand, if  $\log x = -5.45$ , then  $x$  is on the order of  $10^{-6}$ , and we are talking about a few millionth.

### 13.3.2 The Function $f(x) = \log x$

Here we take a brief look at the logarithm from a function perspective, and set<sup>27</sup>

$$f(x) = \log x$$

For every input  $x > 0$  we find exactly one output  $y$ , which can be any real number.

The graph of  $f$  is shown in Figure 27, and it supports that the domain of  $f$  are the positive numbers, and the range are all real numbers.

When we define

$$g(x) = 10^x$$

---

<sup>27</sup>As we said before, there is not much difference between  $y = \log x$  and  $f(x) = \log x$ . The notation has changed, and the point of view is slightly different.

By the way, a calculator responds with "log(" when you use the LOG key, supporting the notion of a function, and when we use  $\log x$  we are talking about a function, whose name comprises three letters, rather than the customary "f" or "g", and it is customary to omit the parentheses.

the identity (10) implies that

$$g(f(x)) = 10^{f(x)} = 10^{\log x} = x$$

while (11) leads to

$$f(g(x)) = \log g(x) = \log 10^x = x$$

Hence, the logarithm and  $10^x$  are inverse functions!

The basic rules for logarithms become

$$\begin{aligned} f(a \cdot b) &= f(a) + f(b) \\ f(a^n) &= n f(a) \end{aligned}$$

in function notation. Moreover,  $f$  has the properties that

$$f(1) = 0 \quad \text{and} \quad f(10) = 1$$

and that

$$f\left(\frac{1}{a}\right) = -f(a)$$

and

$$f\left(\frac{a}{b}\right) = f(a) - f(b)$$

### 13.4 Solving Exponential Equations

The power of logarithms come into play when we want to solve equations of the type

$$a^x = b$$

These are called *exponential equations*, as the unknown is in the exponent.  $6^x = 24$  is an example, and the problem is much different from solving  $6x = 24$ .

**Example:** We solve  $6^x = 24$ . Since the quantities are the same, their respective logarithms must be equal ("take the logarithm on both sides") and we obtain that

$$\log 6^x = \log 24$$

We apply identity (13) to the first expression and then solve for  $x$ :

$$\begin{aligned} x \log 6 &= \log 24 \\ 0.778x &= 1.380 \\ x &= \frac{1.380}{0.778} = 1.774 \end{aligned}$$

And indeed,  $6^{1.774} = 24$ .

There is nothing special about 6 and 24. If we go through the same process for arbitrary (positive) numbers  $a$  and  $b$  we find

$$\begin{aligned} a^x &= b \\ \log a^x &= \log b \\ x &= \frac{\log b}{\log a} \end{aligned}$$

Result:

$$a^x = b \quad \Leftrightarrow \quad x = \frac{\log b}{\log a} \quad (14)$$

**Examples:** The equation  $5^x = 16$  has solution

$$x = \frac{\log 16}{\log 5} = 1.204 ,$$

and  $3^x = 0.1$  has solution

$$x = \frac{\log 0.1}{\log 3} = \frac{-1}{0.447} = -2.096$$

**Logarithms for any base.** Logarithms can be defined with an arbitrary base  $b$ . The common logarithm uses base  $b = 10$ . The most convenient logarithm in Calculus is the *natural logarithm*, which uses Euler's number  $e = 2.7182\dots$  as a base. Computer scientists have use for the logarithm with base 2. The general definition of a logarithm is

$$y = \log_b x \quad \Leftrightarrow \quad x = b^y$$

In this sense  $\log x = \log_{10} x$  and the natural logarithm becomes  $\ln x = \log_e x$ .

The equation  $x = b^y$  can be solved for  $y$  by formula (14), and we find that

$$y = \log_b x = \frac{\log x}{\log b}$$

If you know one logarithm, you know them all!

For example,  $y = \log_2 20$  is the number which solves  $2^y = 20$ . But the solution of this equation, it is  $y = \frac{\log 20}{\log 2} = 4.322$ , and we have just computed that

$$\log_2 20 = 4.322$$

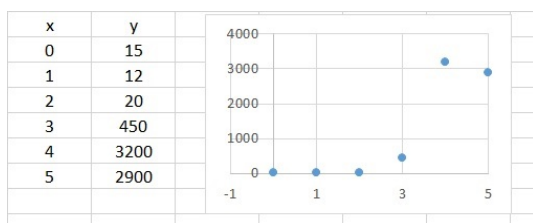
Calculators usually have the LOG key for the common logarithm, and the LN key for the natural logarithm.

## 13.5 Applications

### 13.5.1 Log Plots

We have already looked at logarithmic scales in the graphing chapter. The big advantage of logarithmic scales is that it is possible to show large and small quantities in the same graph in a meaningful way.

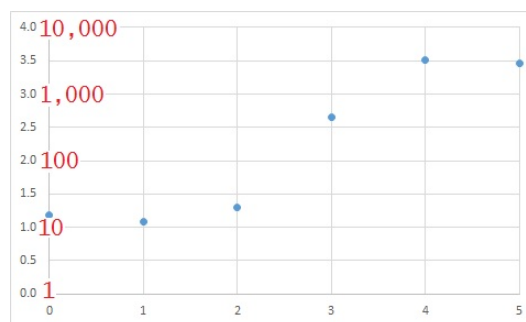
A data set with its scatter plot is shown below.



We see that the details for the small data have been lost, because the large value of 3,200 dominates the graph. The 67% increase from 12 to 20 becomes insignificant when other values lie in the thousands.

To remedy this problem we graph the logarithms of the data.

$x$	$y$	$\log y$
0	15	1.176
1	12	1.079
2	20	1.301
3	450	2.653
4	3,200	3.505
5	2,900	3.462



But the average user does not care for logarithms. So we display  $10^y$  on the  $y$ -axis, and the original magnitudes are restored. This is the basic idea behind graphs on a logarithmic scale.

We will see further applications of this concept in the next chapters.

### 13.5.2 pH Values

The term pH stands for power of hydrogen. If the molar concentration of hydrogen of an aqueous solution is denoted by  $[H^+]$ , then the pH is defined as

$$pH = -\log[H^+] \quad (15)$$

**Example:** At  $25^\circ C$  distilled water has molar concentration

$$[H^+] = \frac{10^{-7} \text{ moles of hydrogen}}{\text{Liter}} = 10^{-7} \text{ M}$$

Therefore the pH of water is

$$pH = -\log 10^{-7} = -(-7) = 7$$

Water is considered to be pH neutral. Solutions with pH values less than 7 are acidic, those with higher pH are alkaline (basic). Be aware that a high pH represents a low hydrogen concentration and vice versa.

We observe, that a concentration must always be less than one, and therefore its logarithm will always be negative (see Figure 27). The purpose of the minus sign in the definition (15) is to make the pH a positive quantity. Commonly, pH values are given with one decimal place accuracy, and there is no need for higher precision.

If the pH is known, we can use formula (15) to calculate the hydrogen concentration as

$$[H^+] = 10^{-pH} \text{ M}$$

**Example:** The pH of coffee is 5.5. Therefore

$$\begin{aligned} [H^+] &= 10^{-pH} \text{ M} = 10^{-5.5} \text{ M} \\ &= 3.2 \cdot 10^{-6} \text{ M} \end{aligned}$$

More examples can be found worked problems below.

### 13.5.3 Moment Magnitude Scale

The strength of earthquakes are commonly given by values on the Richter scale. It is named in honor of C. F. Richter, a physicist at CalTech in Pasadena, CA, in the 1930ies. The measurement is based on the amplitude of the waves shown by a specific seismograph located 100 km from the epicenter of the quake.

The deflection of a needle in a scientific instrument is not a meaningful physical quantity. The energy (moment) released by an earthquake is much more useful. In 1979, Hanks and Kanamori also of CalTech, developed the moment magnitude scale  $M_w$ ; it is defined as

$$M_w = \frac{2}{3} \log m - 10.7$$

Here  $m$  is the moment released by the earthquake measured in dyne-cm (erg), and the constants are chosen in such a way that the values of  $M_w$  resemble those of the Richter scale.

**Example:** The energy of the Tohoku earthquake (Fukushima, March 11, 2011) was  $3.5 \cdot 10^{29}$  dyne-cm. Thus, the moment magnitude was

$$\begin{aligned} M_w &= \frac{2}{3} \log(3.5 \cdot 10^{29}) - 10.7 \\ &= \frac{2}{3} \cdot 29.5 - 10.7 \\ &= 19.7 - 10.7 = 9.0 \end{aligned}$$

The aftershock of the same quake had moment magnitude  $M_w = 7.1$ , and we can use the definition to calculate its energy:

$$\begin{aligned} 7.1 &= \frac{2}{3} \log m - 10.7 \\ 17.8 &= \frac{2}{3} \log m \\ \log m &= \frac{3}{2} \cdot 17.8 = 26.7 \\ m &= 10^{26.7} = 5.0 \cdot 10^{26} \end{aligned}$$

We see that the moment of the aftershock was  $5.0 \cdot 10^{26}$  dyne-cm, which is significantly less than the energy of the original quake (by a factor of about 700).

Sometimes the energy is given in seismic moments  $M$ , which is the equivalent of Nm (Newton-meter) or Joule (J). In this case the formula needs to be adjusted. Since  $1 \text{ J} = 10^7 \text{ dyne-cm}$ , it follows that

$$m = M \cdot 10^7$$

and therefore

$$\log m = \log(M \cdot 10^7) = \log M + 7$$

Substitution yields

$$M_w = \frac{2}{3} \log m - 10.7$$



$$\begin{aligned}
&= \frac{2}{3} (\log M + 7) - 10.7 \\
&= \frac{2}{3} \log M + \frac{14}{3} - 10.7 \\
&= \frac{2}{3} \log M - 6.03
\end{aligned}$$

### 13.6 Worked Problems

1. Without using a calculator determine

- (a)  $\log 10,000,000$
- (b)  $\log 0.1$
- (c)  $\log \frac{1}{100}$

Solutions:

- (a)  $10,000,000 = 10^7$ , therefore  $\log 10,000,000 = 7$
- (b)  $0.1 = 10^{-1}$ , therefore  $\log 0.1 = -1$
- (c)  $\frac{1}{100} = 10^{-2}$ , therefore  $\log \frac{1}{100} = -2$

2. Convert from exponential to logarithmic form, or vice versa.

- (a)  $10^x = 1,000$
- (b)  $10^x = 25$
- (c)  $x = 10^{3.7}$
- (d)  $\log x = -2$
- (e)  $\log x = 3.7$
- (f)  $\log x = \frac{2}{3}$

Solutions: The question just asks for conversion, and giving the values for  $x$  is an added bonus.

- (a)  $x = \log 1,000 (= 3)$
- (b)  $x = \log 25 (= 1.398)$
- (c)  $\log x = 3.7$
- (d)  $x = 10^{-2} (= 0.01)$
- (e)  $x = 10^{3.7} (= 5,011.9)$

$$(f) \ x = 10^{2/3} (= 4.642)$$

3. Estimate  $\log 3$ .

Solution: Here we need a convenient approximation of  $3^n$ , similar to  $2^{10} \approx 1,000$ .

Solution One: We use  $3^4 = 81 \approx 80$  along with the already established result that  $8 \approx 10^{0.9}$ . Then

$$\begin{aligned}
3^4 &= 81 \approx 80 = 10 \cdot 8 \approx 10 \cdot 10^{0.9} \\
&= 10^{1.9}
\end{aligned}$$

Therefore

$$3 = (3^4)^{1/4} \approx (10^{1.9})^{1/4} = 10^{0.475}$$

and thus  $\log 3 \approx 0.475$ . Comparison to the calculator values shows that our result is too low, with a relative error<sup>28</sup> of 0.4%.

Solution Two. We start from scratch without previous estimates. Patient trial and error experiments lead to

$$3^{21} = 1.046 \cdot 10^{10} \approx 10^{10}$$

Therefore,

$$3 \approx (10^{10})^{1/21} = 10^{10/21}$$

and  $\log 3 \approx \frac{10}{21} = 0.4762$ , with relative error 0.2%.

4. Solve for  $x$

- (a)  $100^x = 1,000$
- (b)  $2^x = 0.125$
- (c)  $12^x = \frac{1}{3}$
- (d)  $2^{x+1} = 3^x$

---

<sup>28</sup>  $\frac{\text{estimate} - \text{exact}}{\text{exact}} = \frac{-0.0021}{\log 3} = -0.0044$

Solutions:

- (a) Solution One: The equation has the form  $a^x = b$  with  $a = 100$  and  $b = 1,000$ , and formula (14) implies that

$$x = \frac{\log 1000}{\log 100} = \frac{3}{2} = 1.5$$

Solution Two: This problem can be done without logarithms.

$1,000 = 10^3$ ,  $100 = 10^2$  and  $100^x = (10^2)^x = 10^{2x}$ , and the problem becomes

$$10^{2x} = 10^3$$

Comparing the powers yields  $2x = 3$  and therefore  $x = \frac{3}{2} = 1.5$ .

- (b) With formula (14) we find that

$$x = \frac{\log 0.125}{\log 2} = -3$$

This result is not surprising, because

$$0.125 = \frac{1}{8} = \frac{1}{2^3} = 2^{-3}$$

- (c) Logarithms are required here, and formula (14) implies that

$$\begin{aligned} x &= \frac{\log \frac{1}{3}}{\log 12} = \frac{-\log 3}{\log 12} \\ &= \frac{-0.477}{1.079} = -0.442 \end{aligned}$$

- (d) Take the logarithm on both sides and solve for  $x$ :

$$\begin{aligned} 2^{x+1} &= 3^x \\ \log 2^{x+1} &= \log 3^x \\ (x+1) \log 2 &= x \log 3 \\ 0.301(x+1) &= 0.477x \\ 0.301x + 0.301 &= 0.477x \\ 0.301 &= 0.176x \\ x &= \frac{0.301}{0.176} = 1.710 \end{aligned}$$

Graphical solutions are always an option, either to find solutions or to confirm known results. Plots for all four parts are given in Figure 28.

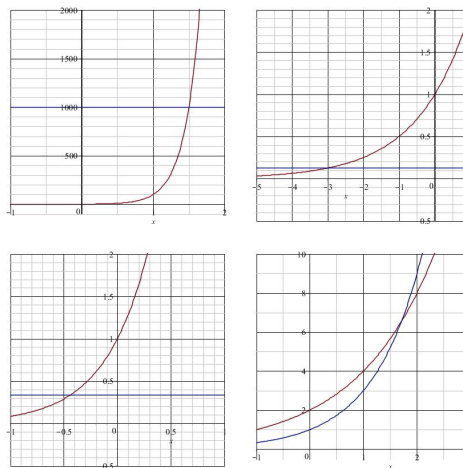


Figure 28: Graphs for Problem 4

5. What is the pH of blood, given that the hydrogen concentration is  $4 \cdot 10^{-8}$  M?

Solution: Apply the definition (15) to obtain

$$pH = -\log(4 \cdot 10^{-8}) = -(-7.4) = 7.4$$

6. For a vinegar we find that  $[H^+] = 7.5 \cdot 10^{-3}$ . What is the pH?

Solution:

$$pH = -\log(7.5 \cdot 10^{-3}) = 2.1$$

7. What is the hydrogen concentration of milk, given that its pH is 6.4?

Solution: Solve for  $[H^+]$  in the definition.

$$\begin{aligned} [H^+] &= 10^{-pH} \text{ M} = 10^{-6.4} \text{ M} \\ &= 4.0 \cdot 10^{-7} \text{ M} \end{aligned}$$

8. The pH of a beer is 4.0. Is its hydrogen concentration higher or lower than that of water?

Solution: A smaller pH means a higher hydrogen concentration. Since the pH values differ by 3, their hydrogen concentrations differ  $1,000 = 10^3$ .

If you don't trust this line of reasoning, you can calculate hydrogen concentration directly:

$$[H^+] = 10^{-4} \text{ M}$$

$10^{-4}$  is 1,000 times larger than  $10^{-7}$ .

9. You come across 40 mL of an unknown mixture and you detect that it contains  $3 \cdot 10^{-5}$  moles of hydrogen. What is the pH? Is it alkaline or acidic?

Solution: As a first step we have to find the molar concentration:

$$\begin{aligned} & \frac{3 \cdot 10^{-5} \text{ moles}}{40 \text{ mL}} \times \frac{25}{25} \\ = & \frac{7.5 \cdot 10^{-4} \text{ moles}}{1,000 \text{ mL}} \\ = & \frac{7.5 \cdot 10^{-4} \text{ moles}}{\text{Liter}} \\ = & 7.5 \cdot 10^{-4} \text{ M} \end{aligned}$$

Now we calculate the pH

$$pH = -\log(7.5 \cdot 10^{-4}) = 3.1$$

and we see that this mixture is acidic.

10. What is the moment magnitude of an earthquake of strength  $10^{20}$  J?

Solution: First we convert to dyne-cm:

$$\begin{aligned} 10^{20} \text{ J} &= 10^{20} \cdot 10^7 \text{ dyne-cm} \\ &= 10^{27} \text{ dyne-cm} \end{aligned}$$

Now we apply the definition

$$\begin{aligned} M_w &= \frac{2}{3} \log 10^{27} - 10.7 \\ &= \frac{2}{3} \cdot 27 - 10.7 \\ &= 18 - 10.7 = 7.3 \end{aligned}$$

11. On Feb. 17, 2015, an earthquake with magnitude  $M_w = 2.4$  was reported near Bluefield, WV. How much energy was released?

Solution: We set  $M_w = 2.4$  and solve the equation for  $m$ .

$$\begin{aligned} 2.4 &= \frac{2}{3} \log m - 10.7 \\ 13.1 &= \frac{2}{3} \log m \\ \log m &= \frac{3}{2} \cdot 13.1 = 19.65 \\ m &= 10^{19.65} = 4.5 \cdot 10^{19} \end{aligned}$$

The moment was  $4.5 \cdot 10^{19}$  dyne-cm.

12. We are comparing two earthquakes. One releases ten times as much energy as the other. How do the moment magnitudes differ?

Solution: Let us say that the smaller quake has moment  $m$  and the larger has moment  $m' = 10m$ . Then the moment magnitude of the bigger earthquake is

$$\begin{aligned} M'_w &= \frac{2}{3} \log m' - 10.7 \\ &= \frac{2}{3} \log 10m - 10.7 \\ &= \frac{2}{3} (1 + \log m) - 10.7 \\ &= M_w + \frac{2}{3} = M_w + 0.666, 666, \dots \end{aligned}$$

that is, the moment magnitude of the stronger earthquake is about 0.7 units bigger.

### 13.7 Exercises

- Without using a calculator determine
  - $\log 10,000$
  - $\log 0.000,001$
  - $\log \frac{1}{1,000}$
- Convert from exponential to logarithmic form, or vice versa.
  - $10^x = 1.6$
  - $\log x = 1.6$
- Estimate  $\log 125$  and  $\log 1.25$
- Solve for  $x$ :
  - $10^x = 10,000$
  - $10^x = 0.000,1$
  - $10^x = 5$
  - $10^x = 0.2$
  - $\log x = 3$
  - $\log x = -2$
  - $\log x = 0.4$
  - $\log x = 1.4$
  - $5^x = 20$
  - $1.08^x = 2$
  - $2^{x+2} = 4^x$
  - $\log x + \log 4 = \log 40$
  - $\log(x-2) + \log(x+2) = \log 32$
- The pH of Sprite is 3.3. What is its hydrogen concentration?
- The hydrogen concentration of blood is  $[H^+] = 4.0 \times 10^{-8}M$ . What is its pH?
- The pH of milk is 6.4. Is the hydrogen concentration of milk higher or lower than that of water? And by what factor?

- You discover 3L of an unknown solution and you measure a total of  $8 \times 10^{-7}$  moles of  $H^+$  in this mixture. Calculate the pH.
- You change a mixture so that its hydrogen concentration is doubled. How will the pH change?
- The 1906 San Francisco earthquake had moment magnitude 7.9. How much energy was released by the quake?
- What is the moment magnitude  $M_w$  of an earthquake when the moment is  $3.9 \times 10^{27}$  dyne-cm?

### Answers

- 4
  - 6
  - 3
- $x = \log 1.6$
  - $x = 10^{1.6}$
- 2.1 and 0.1, respectively.
- 4
  - 4
  - 0.6990
  - 0.6990
  - 1,000
  - 0.01
  - 2.5119
  - 25.119
  - 1.8614
  - 9.006
  - 2
  - 10
  - 6

5.  $5 \times 10^{-4}$  M
6. 7.4
7.  $4 \times 10^{-7}$ ; the  $H^+$  concentration of milk is four times higher.
8.  $6.574 \approx 6.6$
9. The pH will decrease by 0.3, the mixture is more acidic.
10.  $7.943 \times 10^{27}$  dyne-cm
11. 7.7

## 14 Exponential Functions

In everyday language we often attach the term *exponential* when things change rapidly. In an inflation prices increase exponentially, in an epidemic a disease grows exponentially, kudzu is spreading exponentially, and so on. In mathematics, this term is reserved for a very special setting, which is the topic of this section.

Population models are arguably the most prominent biological application of exponential growth; this applies to easily observed and counted animals, as well as bacteria and other cell growth. Radioactive decay is the prime example of exponential decay.

### 14.1 Introduction

Here we introduce the basic concepts of growth rates and multipliers in a hypothetical example.

Suppose that the population of monkeys on an island increases by 6% annually. Beginning with 455 animals in 2015, predict the population for future years, and in particular, estimate the population in 2050.

We begin by calculating the population in 2016. 6% of 455 equals 27.3, and we expect 482.3 monkeys on the island in the following year. In terms of mathematics, the calculation looks like

$$\begin{aligned} 482.3 &= 455 + 0.06 \cdot 455 \\ &= (1 + 0.06) \cdot 455 \\ &= 1.06 \cdot 455 \end{aligned}$$

In order to derive a pattern it helps tremendously to factor the term 1.06. When we look at the population for 2017, we have to repeat the calculation with 455 being replaced by 482.3.

$$\begin{aligned} &482.3 + 0.06 \cdot 482.3 \\ &= (1 + 0.06) \cdot 482.3 \end{aligned}$$

$$\begin{aligned}
&= 1.06 \cdot 482.3 = 1.06 \cdot (1.06 \cdot 455) \\
&= 1.06^2 \cdot 455 = 511.2
\end{aligned}$$

and the population in 2017 should be around 511 animals.

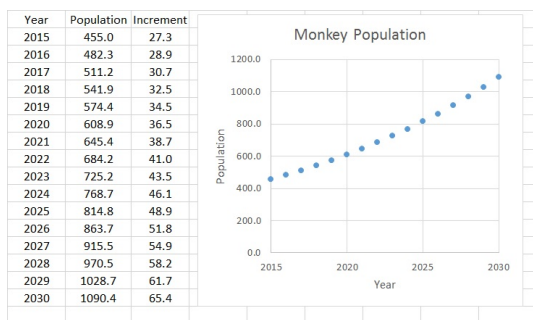
We notice that by stepping from one year to the next, the population is multiplied by 1.06 each time. One year after 2016, the population will be  $1.06 \cdot 455$ , two years after 2015 it will be  $1.06^2 \cdot 455$ , and in 2050, which is 35 years after 2015, we expect the population to be at  $1.06^{35} \cdot 455 = 3,497$  monkeys.

The formula of the population  $x$  years after 2015 becomes

$$y = 455 \cdot 1.06^x$$

There are three important quantities involved. First, we need to know the initial population  $y_0$ , which was 455 in the example. Secondly, we need to know the growth rate  $r$ , which was 0.06=6%. And finally, there is a multiplier  $M$ , which in our case was  $M = 1.06$ . The multiplier and the growth rate are related by  $M = 1 + r$ .

In our computations we allow decimal values in population numbers. After all, this is just a model. Round to the nearest integer if 482.3 monkeys bothers you. An EXCEL simulation for the first 15 years is shown below



## 14.2 Exponential Growth and Decay

An exponential model has the form

$$y = y_0 M^x = y_0 (1 + r)^x \quad (16)$$

Here the variables are

- $x$  independent variable, usually time
- $y$  dependent variable, often population
- $y_0$  initial value
- $r$  growth or decay rate,  $r > -1$
- $M$  multiplier  $M = 1 + r$ ,  $M > 0$

In the introductory example we had  $y_0 = 455$ ,  $r = 0.06$  and  $M = 1.06$ .

Our model covers exponential decay as well, in which case  $r$  is negative. But a decline cannot be faster than 100%, and if the decline equals 100%, everything vanishes immediately, and a mathematical model is no longer needed. For this reason we require that  $r > -1$ .

The multiplier and the rate are related by

$$M = 1 + r \Leftrightarrow r = M - 1$$

and  $r > -1$  implies that  $M > 0$ . Multipliers must be positive!  $r = 0$  is equivalent to  $M = 1$ , and in this case the quantity of interest  $y$  remains constant.

For  $x = 0$  the formula (16) reduces to

$$y = y_0 M^0 = y_0$$

This is why  $y_0$  is called the initial condition. It is the value of  $y$  when  $x = 0$ .

**Example:** A fish population increases from 357 to 411 fish within one year. We assume that the population continues to grow the same rate, and we want to know the population five years later.

The information can be summarized as

$x$	$y$
0	357
1	411

and since 357 is associated with  $x = 0$  it follows that  $y_0 = 357$ . If we substitute the data for  $x = 1$  into the definition (16), we find that

$$411 = 357 \cdot M^1 = 357M$$

and thus

$$M = \frac{411}{357} = 1.151$$

This means that in order to get from one year to the next, we have to multiply the population by 1.151. If we want to look five years ahead, we have to multiply by  $M$  five times, and our prediction is

$$357 \cdot 1.151^5 = 722.0 \text{ fish}$$

The general model is

$$y = 357 \cdot 1.151^x$$

Knowing  $M$  means that we also know the growth rate:

$$r = M - 1 = 0.151 = 15.1\%$$

In this example  $r = 15.1\%$ , and we observe that our population about doubled in size during this five year period.

### Linear versus Exponential Growth.

There is a huge difference between linear and exponential models. In linear models the quantity of interest  $y$  changes in *fixed increments*, and this change is responsible for the slope. In exponential models the quantity  $y$  changes at a *fixed rate*, which leads to the multiplier  $M = 1 + r$ .

**Example:** We return to the fish population, which increased from 357 to 411 fish from one year to the next.

In the exponential model we argued that the change represented a 15.1% increase, that the population will continue to grow at this rate, which in turn resulted in the equation

$$y = 357 \cdot 1.151^x$$

For a linear model on the other hand, we would argue that the number of fish increased

by 54 fish, and that each year we would have additional 54 fish, which results in the model

$$y = 357 + 54x$$

The two models agree for  $x = 0$  and for  $x = 1$ , other than that they are much different, especially when  $x$  becomes large.

A side by side comparison of the two models is given in Figure 29.

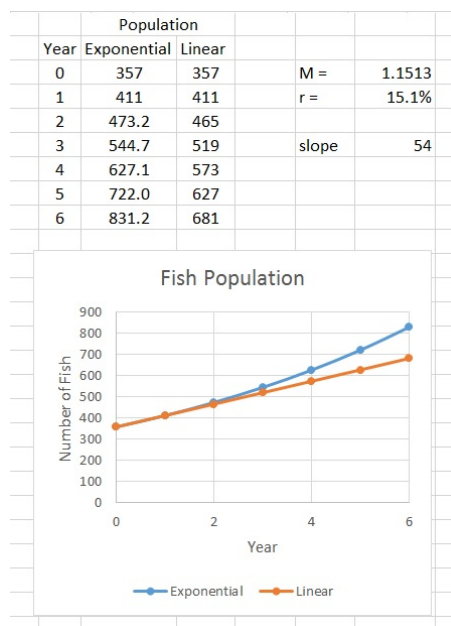


Figure 29: Linear vs. Exponential

**Graphs of Exponential Functions** have a very characteristic shape. The plots for  $y = 48 \cdot 1.2^x$  and  $y = 75 \cdot 0.8^x$  are shown in Figure 30 below. The growth curve increases toward infinity, while the decay curve approaches the  $x$ -axis as  $x$  increases. Notice that the curves look somewhat alike, with the  $x$ -directions reversed.

When exponential functions are graphed on a logarithmic scale they become linear! Recall<sup>29</sup> that graphs on a logarithmic scale are

<sup>29</sup>This process was outlined in Section 13.5.1.

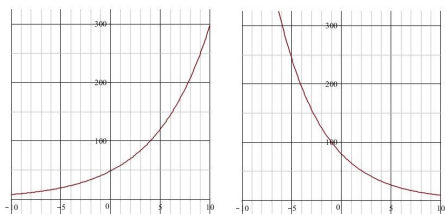


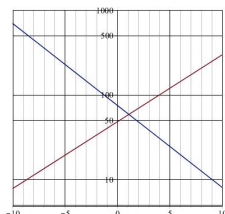
Figure 30:  $y = 48 \cdot 1.2^x$  and  $y = 75 \cdot 0.8^x$

just a plot the logarithms of the expressions, along with a manipulation of the vertical axis. In our case we find that

$$\begin{aligned}\log y &= \log(y_0 \cdot M^x) \\ &= \log y_0 + \log M^x \\ &= x \log M + \log y_0\end{aligned}$$

But this is an equation of a line with slope  $m = \log M$  and  $y$ -intercept  $b = \log y_0$ .

The graphs of  $y = 48 \cdot 1.2^x$  and  $y = 75 \cdot 0.8^x$  on a logarithmic scale are shown below.



The notion of converting exponential curves into lines via logarithms is exploited in the approximation of almost exponential data.

**Exponentials in Function Notation.** We can also write

$$f(x) = y_0 M^x$$

and express the exponential growth model as a function. It satisfies

$$f(0) = y_0$$

which confirms that  $y_0$  is the initial value (associated with  $x = 0$ ). Moreover

$$\begin{aligned}f(x+1) &= y_0 M^{x+1} = y_0 M^x M \\ &= M f(x)\end{aligned}$$

and we see that stepping forward by one time unit is equivalent to multiplication by  $M$ . Stepping forward by  $T$  time units is equivalent to multiplication by  $M^T$  because

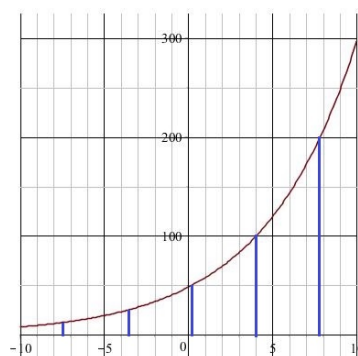
$$\begin{aligned}f(x+T) &= y_0 M^{x+T} = y_0 M^x M^T \\ &= M^T f(x)\end{aligned}$$

We will use this principle when we determine doubling time and half-life ( $M^T = 2$  and  $M^T = \frac{1}{2}$ , respectively).

### 14.3 Doubling Time

We begin with an example and look at the exponential function

$$y = 48 \cdot 1.2^x$$



The graph shows the  $x$  values for  $y = 12.5$ ,  $y = 25$ ,  $y = 50$ ,  $y = 100$  and  $y = 200$ , and we see these points are about equally spaced along the  $x$ -axis. This tells us that doubling the  $y$ -value always takes the same amount of time.

We can also observe this fact numerically.

$x$	$48 \cdot 10^x$
0	48
1	57.3
2	69.1
3	82.9
4	99.5
5	119
6	143



When  $x = 0$  we have  $y = 48$ , and  $y$  reaches  $96 = 2 \cdot 48$  a little bit before  $x = 4$ . But we can also start with  $x = 2$ , where  $y \approx 70$  and when  $x = 6$  we have already passed the  $y = 140$  mark. Again, it took a little less than four time units to double the  $y$ -value.

We can determine the doubling time analytically. At  $x = 0$  we have  $y = 48$  and we need to find  $x$  such that

$$96 = 48 \cdot 1.2^x$$

Division by 48 results in

$$1.2^x = 2$$

which is an exponential equation. Fortunately, we have already discussed equations of this form in Section 13.4, and formula (14) shows that

$$x = \frac{\log 2}{\log 1.2} = 3.802$$

which confirms our observation of a doubling time a little less than four.

We move on and we seek to find the doubling time  $T$  for the general exponential model

$$y = y_0 \cdot M^x$$

We pick any  $x$ , then at time  $x + T$  the  $y$ -value should have doubled. This translates into

$$y_0 M^{x+T} = 2y_0 M^x$$

We divide by  $y_0$ , which shows that the starting values are immaterial in this context, and obtain

$$M^{x+T} = M^x M^T = 2 M^x$$

Further division by  $M^x$  results in

$$M^T = 2 \quad (17)$$

which relates the doubling time  $T$  and the multiplier  $M$ , and we find that

$$T = \frac{\log 2}{\log M} \quad (18)$$

**Examples:** The doubling time for the growth model  $y = 25 \cdot 1.05^x$  is

$$T = \frac{\log 2}{\log 1.05} = 14.2$$

and that for  $y = 5^x$  is

$$T = \frac{\log 2}{\log 5} = 0.431$$

**Example:** A bacteria culture grows exponentially and it increased from 20 bacteria initially to 1,000 bacteria within two hours.

Let us first estimate the doubling time, and then compute it analytically.

Doubling once results in 40 bacteria, doubling again leads to 80 bacteria and so on. Doubling five times results in

$$20 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 20 \cdot 2^5 = 640$$

bacteria, and doubling one more time yields 1,280 bacteria. Thus, our bacteria population doubles between 5 and 6 times within two hours, and cell divisions should occur about every 20 to 24 minutes<sup>30</sup>.

The string of inequalities

$$2^5 = 32 \leq \frac{1,000}{20} = 50 \leq 64 = 2^6$$

is another way to approach this problem. The population increased by a factor of 50, which requires between 5 and 6 duplications.

For the analytic solution we measure time  $x$  in hours. Then the model becomes

$$y = 20 \cdot M^x$$

and substitution of  $x = 2$  and  $y = 1,000$  results in

$$1,000 = 20 \cdot M^2$$

---

<sup>30</sup>We have  $\frac{2 \text{ hours}}{6} = \frac{1 \text{ hour}}{3} = 20 \text{ min}$  and  $\frac{2 \text{ hours}}{5} = 0.4 \text{ hours} = 24 \text{ min}$ .

and we deduce that

$$M = \sqrt{\frac{1000}{20}} = \sqrt{50}$$

The formula for the doubling time (18) then implies that

$$T = \frac{\log 2}{\log \sqrt{50}} = 0.352$$

Conversion to minutes shows a doubling time of 21.3 minutes.

**Example (Rule of 72):** This rule is used in business to estimate the time it takes to double an investment at a given interest rate. It is a rule of thumb:

$$\text{Interest Rate} \times \text{Doubling Time} = 72$$

At 3% interest it will take an investment 24 years to double in value, because  $3 \times 24 = 72$ . If an investment was doubled in 8 years, the interest rate was 9% ( $9 \times 8 = 72$ ), and so on. This is an approximate rule, because if an investment doubles just in one year, the interest rate is 100% and not 72%.

Compound interest follows the exponential model (16) as well. \$455 invested at 6% interest uses the same mathematics as monkey population of 455 animals growing at a 6% annual rate, and we can use the Rule of 72 to estimate doubling times in population examples as well.

In the opening example we started with 455 monkeys in 2007, and in the simulation it took until 2027 for the population to reach 915.5. Doubling took about 12 years at 6% growth rate, as predicted by the Rule of 72.

In the fish population example we had a growth rate of 15.1%, and the Rule of 72 estimates a doubling time of a little under five years ( $T \approx 4.8$ ), which is consistent with the data in Figure 29.

In the example with  $y = 48 \cdot 1.2^x$ , we have  $M = 1.2$  and hence  $r = M - 1 = 0.2 = 20\%$ . The Rule of 72 estimates a doubling time of  $72/20 = 3.6$ , which is close to the exact value  $T = 3.802$  with an error margin of 6%.

How does this rule work?

With the tools of Calculus it can be shown that

$$\log(1 + r) \approx 0.434r$$

is a very good approximation for small numbers  $r$ . But  $1 + r = M$ , and the formula (18) for  $T$  implies that

$$\begin{aligned} \log 2 &= T \cdot \log M \\ &= T \cdot \log(1 + r) \approx 0.434rT \end{aligned}$$

Therefore,

$$r \cdot T \approx \frac{\log 2}{0.434} = 0.693$$

In the Rule of 72 we entered the percentages directly (8% as 8 and not as 0.08 as we normally do), and thus

$$100 r \cdot T \approx 69.3$$

This doesn't make for a good rule. But change 69.3 to 72, which is divisible by 2, 3, 4, 6, 8, 9, 12, 18, 24 and 36, and you have a convenient estimation tool!

There is an alternate, equivalent way to express exponential growth using the doubling time. In (17) we established that  $M^T = 2$ . Thus,

$$\begin{aligned} M &= 2^{1/T} \\ M^x &= 2^{x/T} \\ y &= y_0 M^x = y_0 2^{x/T} \end{aligned}$$

and we have

$$y = y_0 2^{x/T} \quad (19)$$

The multiplier no longer appears explicitly, but it can be recovered from  $M = 2^{1/T}$ .

**Example:** A lung cancer cell doubles by mitosis about every three months. Single cancer cells cannot be detected by X-rays and it takes about one billion cells for a tumor to become visible. Beginning with a single cell we want to determine how long it will take until the cancer can be detected by X-rays. This example is adapted from Langkamp/Hull.

We measure time in years. Then the doubling time is  $T = 0.25 = 1/4$ , and using formula (19) our model for the number of tumor cells becomes

$$y = 2^{x/0.25} = 2^{4x} = (2^4)^x = 16^x$$

since  $y_0 = 1$ . We now have to set  $y = 10^9$  and solve the resulting exponential equation.

$$\begin{aligned} 16^x &= 10^9 \\ x &= \frac{\log 10^9}{\log 16} = \frac{9}{1.204} \approx 7.5 \end{aligned}$$

and we see that it takes about seven and a half years until the tumor is visible.

**Generation Time.** For cell (bacterial) growth the time between cell divisions is called generation time (time per generation), and it is the equivalent of the doubling time. If we denote the number of generations by  $n$ , the total elapsed time becomes  $x = nT$ .

**Example:** If the generation time is  $T = 24$  minutes, and we go through  $n = 8$  generations, then the total time is  $x = 24 \cdot 8 = 192$  minutes, or 3 hours and 12 minutes.

If, on the other hand, we observe cell growth for two hours (120 minutes), and we know that the time per generation is 32 minutes, then we have gone through  $n = \frac{x}{T} = \frac{120}{32} = 3.75$  generations.

With this terminology we have  $x = nT$ , and thus  $x/T = n$  and the exponential model (19) simplifies to

$$y = y_0 \cdot 2^n$$

**Example:** We review the bacterial growth from above: 20 bacteria grew to 1,000 bacteria in two hours. Since initially  $y_0 = 20$  bacteria are present, the model becomes

$$y = 20 \cdot 2^n$$

Setting  $y = 1,000$ , we find that

$$\begin{aligned} 1,000 &= 20 \cdot 2^n \\ 2^n &= \frac{1,000}{20} = 50 \\ n &= \frac{\log 50}{\log 2} = 5.644 \end{aligned}$$

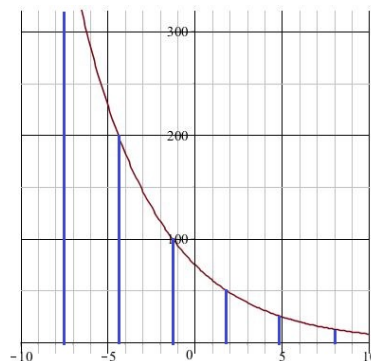
and we see that we went through 5.644 generations in 2 hours, which accounts for a generation time of

$$T = \frac{120 \text{ minutes}}{5.644} = 21.262 \text{ min}$$

which confirms the result we found before.

## 14.4 Half-Life

The notion of doubling time is linked to exponential growth. Its counterpart for exponential decay is half-life. It is the time it takes for a substance to reduce to one half of its original size.



The graph shows the curve  $y = 75 \cdot 0.8^x$ , and we marked off the points corresponding to  $y = 400$ ,  $y = 200$ ,  $\dots y = 12.5$ . The spacing between the points is uniform along the  $x$ -axis, about 3.1 units apart, and we call this the half-life.

We still work with the exponential function  $y = y_0 M^x$ , where  $M < 1$ , resulting in the decay. We denote the half-life by  $H$ ; then value of  $y$  at time  $x+H$  must be one half of the value at time  $x$ . We translate this statement into an equation, which we will then solve for  $H$ :

$$\begin{aligned} y_0 M^{x+H} &= \frac{1}{2} y_0 M^x \\ M^x M^H &= \frac{1}{2} M^x \\ M^H &= \frac{1}{2} \\ H &= \frac{\log 1/2}{\log M} = -\frac{\log 2}{\log M} \end{aligned}$$

and we have a formula for  $H$ .  $M < 1$ , and therefore its logarithm is negative, and the minus in the formula makes  $H$  a positive quantity.

As we did for the doubling time, we can use the half-life in an alternative description for exponential decay:

$$y = y_0 \cdot 2^{-x/H} \quad (20)$$

**Example:** Let  $y = 75 \cdot 0.8^x$ . Here  $M = 0.8$  and the half-life becomes

$$H = -\frac{\log 2}{\log 0.8} = -\frac{0.301}{-0.969} = 3.106$$

If we want to express  $y$  in the alternate form (20), it is easiest to look at the exponent first:

$$\frac{x}{H} = \frac{x}{3.106} = 0.322x$$

and thus

$$y = 75 \cdot 2^{-0.322x}$$

The reader is encouraged to plot  $y = 75 \cdot 0.8^x$  and  $y = 75 \cdot 2^{-0.322x}$  in the common graph to confirm that they completely overlap. The connection between the curves results from  $2^{-0.322} = 0.8$ .

**Example (Radio Carbon Dating):** The abundance of the radioactive isotope  $^{14}\text{C}$  is about one part per one trillion  $^{12}\text{C}$  isotopes. All living organisms exhibit this ratio. But when an organism dies, the radioactive isotope decays with a half-life of 5,730 years, and the ratio becomes smaller.

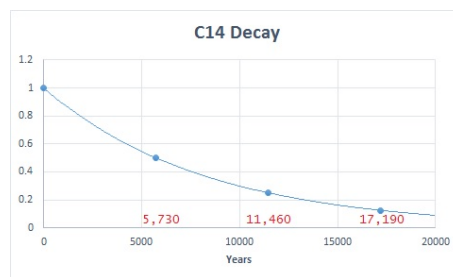


Figure 31: Decay of  $^{14}\text{C}$

After 5,730 years only one half of the natural  $^{14}\text{C} : ^{12}\text{C}$  ratio is present, after another 5,730 years (11,460 years from the beginning) the ratio is down to one quarter of the natural ratio, and so on, and the ratio after  $x$  years can be described by

$$y = 2^{-x/5,730}$$

Scientists can measure the  $^{14}\text{C} : ^{12}\text{C}$  ratio of artifacts, and they use it for age determination.

Suppose that, for instance, a piece of wood contains only 20% of the natural  $^{14}\text{C} : ^{12}\text{C}$  ratio. Then substitution results in

$$0.2 = 2^{-x/5,730}$$

and we find that (exponential equation method)

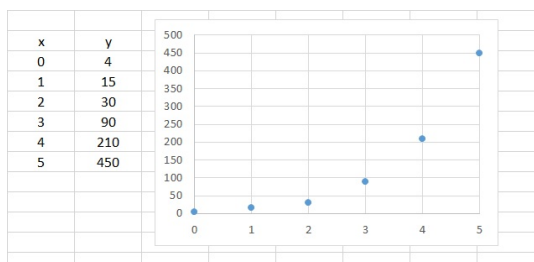
$$-\frac{x}{5,730} = \frac{\log 0.2}{\log 2} = -2.322$$

$$\begin{aligned}x &= (-2.322) \times (-5,730) \\ &= 13,304\end{aligned}$$

and we conclude that the piece of wood is about 13,300 years old. This result agrees well with the graph in Figure 31.  $y = 0.2$  occurs a little past the half way mark between 10,000 and 15,000 years.

## 14.5 Almost Exponential Data

In biological observations, the collected data may indicate an exponential trend, but the fit is not perfect. In this case we are looking for an approximating exponential curve. We explain the method by example.



The data and their graph indicate an exponential trend.

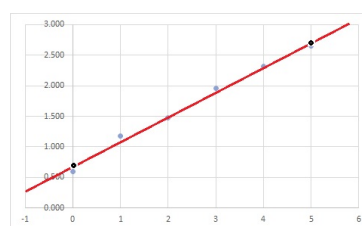
In order to find an approximating curve we take advantage of the fact that exponential functions become linear when we take logarithms. This leads to the following procedure:

1. Take the logarithms of the  $y$ -values.
2. Carefully graph  $x$  against the logarithms.
3. Find an approximating line by the straight-edge method (see Section 12.3).
4. Convert the approximating line to exponential form.

We continue with the example and take logarithms.

$x$	$y$	$\log y$
0	4	0.602
1	15	1.176
2	30	1.477
3	90	1.954
4	210	2.322
5	450	2.653

A graph of the logarithmic data with an approximating line are shown below.



We select the points (0, 0.7) and (5, 2.7) for the approximating line. Then

$$m = \frac{2.7 - 0.7}{5 - 0} = \frac{2}{5} = 0.4$$

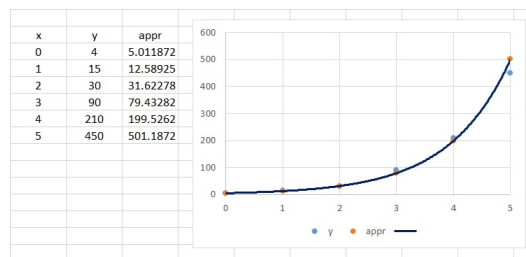
But we are modeling the logarithm of the data, and the "line" becomes

$$\log y = 0.4x + 0.7$$

Finally, we convert back to exponential form:

$$\begin{aligned}y &= 10^{\log y} = 10^{0.7+0.4x} \\ &= 10^{0.7} \cdot (10^{0.4})^x = 5.012 \cdot 2.512^x\end{aligned}$$

The graph of the data with the exponential approximation are shown below.



Exponential regression is important and most graphing calculators or software have built-in functions for this task.

In EXCEL plot the data and ask for an for an "Exponential" trendline. There is an option to display the equation. In our example EXCEL would give the exponential approximation as

$$y = 4.8446 e^{0.9323x}$$

The number  $e$  is Euler's constant, and  $e^{0.9323} = 2.540$ , and the EXCEL approximation is equivalent to

$$y = 4.845 \cdot 2.540^x$$

Our eye-balled estimation  $y = 5.012 \cdot 2.512^x$  is not far from the computer result!

On a graphing calculator the data have to be written to lists using the STAT key, followed by EDIT. Once they are entered, use STAT followed by CALC and ExpReg. The calculator should respond with  $y = a * b \wedge x$ , where  $a = 4.8446$  and  $b = 2.540$ . The details may vary with the brand and model of the calculator, but the resulting approximation will be identical to the one found in EXCEL, as all use the least squares regression technique.

## 14.6 Worked Problems

- Find an exponential model of the form  $y = y_0 M^x$  for the data below, and complete the tables. What are  $M$  and  $r$  in each case?

(a)

$x$	$y$
0	20
1	25
2	
3	
4	

(b)

$x$	$y$
0	10
1	
2	
3	
4	50

(c)

$x$	$y$
0	500
1	
2	400
3	
4	

Solutions:

- (a) When  $x = 0$  we have  $y = 20$ . Therefore  $y_0 = 20$ . We also see that

$$\begin{aligned} 25 &= 20M \\ \Rightarrow M &= \frac{25}{20} = 1.25 \end{aligned}$$

and we have found the multiplier. For  $r$  we obtain

$$r = M - 1 = 0.25 = 25\%$$

and we use

$$y = 20 \cdot 1.25^x$$

to complete the table.

$x$	$y$
0	20
1	25
2	31.25
3	39.06
4	48.83

- (b) Again, we have the initial information that  $y_0 = 10$  when  $x = 0$ . We also have a value when  $x = 4$ , and substitution into (16) yields

$$\begin{aligned} 50 &= 10 \cdot M^4 \\ M^4 &= \frac{50}{10} = 5 \\ M &= \sqrt[4]{5} = 5^{1/4} = 1.495 \end{aligned}$$

It follows that

$$r = M - 1 = 0.495 = 49.5\%$$

The model becomes

$$y = 10 \cdot 1.495^x$$

and the completed table takes the form

$x$	$y$
0	10
1	14.95
2	22.36
3	33.44
4	50

- (c) This is very similar to the previous part.  $y_0 = 500$  and

$$\begin{aligned} 400 &= 500 \cdot M^2 \\ M^2 &= \frac{400}{500} = 0.8 \\ M &= \sqrt{0.8} = 0.894 \end{aligned}$$

The values are decreasing and the rate is negative.

$$r = M - 1 = -0.106 = -10.6\%$$

The formula for  $y$  is

$$y = 500 \cdot 0.894^x$$

and the completed table follows.

$x$	$y$
0	500
1	447
2	400
3	358
4	320

- In the year 1890 Shakespeare enthusiasts released 60 European starlings in New York's Central Park. A century later the

starling population in North America was estimated at 200 million (stanford.edu). Assuming the accuracy of the data and an exponential model, estimate year when the population reaches ten billion.

Solution: We use  $x$  to count the years after 1890. Then  $y_0 = 60$  and for  $x = 100$  we have

$$200,000,000 = 60 \cdot M^{100}$$

Now we solve this equation for  $M$

$$\begin{aligned} M^{100} &= \frac{200,000,000}{60} \\ M &= \left( \frac{200,000,000}{60} \right)^{0.01} \\ &= 1.162 \end{aligned}$$

and the exponential model becomes

$$y = 60 \cdot 1.162^x$$

Finally, we set  $y = 10$  billion and solve the resulting exponential equation for  $x$ .

$$\begin{aligned} 10^{10} &= 60 \cdot 1.162^x \\ 1.162^x &= \frac{10^{10}}{60} = 1.67 \cdot 10^8 \\ x &= \frac{\log(1.67 \cdot 10^8)}{\log 1.162} \\ &= \frac{8.222}{0.0652} = 126 \end{aligned}$$

Therefore, the population will reach ten billion 126 years after 1890, which makes it the year 2016.

- Elephantdatabase.org reports that the elephant population in Africa declined from 474,134 animals in 2007 to 421,955 elephants in 2012. Assuming an exponential decay model, predict when the elephant population will drop below 350,000 animals.

Solution: The data are five years apart, therefore the multiplier satisfies

$$\begin{aligned} 421,955 &= 472,134M^5 \\ M^5 &= \frac{421,955}{472,134} = 0.8937 \\ M &= 0.8937^{1/5} = 0.9778 \end{aligned}$$

$r = M - 1 = -0.0222$ . This tells us that the population declines at an annual rate of 2.2%. The population model becomes

$$P = 472,134 \cdot 0.9778^x$$

where  $x$  counts the years after 2007. Setting  $P = 350,000$  yields

$$\begin{aligned} 350,000 &= 472,134 \cdot 0.9778^x \\ 0.9778^x &= \frac{350,000}{472,134} = 0.7413 \\ x &= \frac{\log 0.7413}{\log 0.9778} \\ &= \frac{-0.1300}{-0.009760} = 13.32 \end{aligned}$$

The model predicts that the population drops below 350,000 in the year 2020 (13 years after 2007).

4. (a) A population grows according to the law  $y = 450 \cdot 1.1^x$ . What is the doubling time?
- (b) What is the growth rate for a population which doubles in four days?

Solutions:

- (a) Here  $M = 1.1$ , and formula (18) yields

$$T = \frac{\log 2}{\log 1.1} = 7.27$$

Note that  $r = 0.1 = 10\%$ , and that the Rule of 72 estimates a doubling time of 7.2.

- (b) We know that  $T = 4$ , and we have to find  $M$  first, and then determine  $r$ . It follows from (17) that

$$M = 2^{1/T} = 2^{1/4} = 1.189$$

and thus  $r = M - 1 = 0.189 = 18.9\%$ . This is a daily growth rate. The Rule of 72 predicts  $r = \frac{72}{4} = 18\%$ .

5. A culture of bacteria grows from 200 to 15,000 bacteria in three hours. How many generations have evolved and what is the generation time?

Solution: We use the model

$$y = 200 \cdot 2^n$$

Setting  $y = 15,000$  we find that

$$\begin{aligned} 15,000 &= 200 \cdot 2^n \\ 2^n &= \frac{15,000}{200} = 75 \\ n &= \frac{\log 75}{\log 2} = 6.229 \end{aligned}$$

Thus, we have gone through 6.229 generations in 3 hours (180 minutes), and the time per generation is

$$T = \frac{180 \text{ min}}{6.229} = 28.9 \text{ min}$$

6. Iodine-123 is a radioactive isotope which is used in thyroid therapy. It has a half-life of 13.22 hours. If 2 mg are being used, then how long will it take until only 0.1 mg is present?

Solution: We use the alternate form (20) of the exponential model, and the amount  $y$  of iodine-123 in the body after  $x$  hours is

$$y = 2 \cdot 2^{-x/13.22}$$



The question asks for the time  $x$  such that  $y = 0.1$ . Substitution results in

$$\begin{aligned} 0.1 &= 2 \cdot 2^{-x/13.22} \\ 2^{-x/13.22} &= 0.05 \\ -\frac{x}{13.22} &= \frac{\log 0.05}{\log 2} = -4.322 \\ x &= (-13.22) \times (-4.322) \\ &= 57.1 \end{aligned}$$

Thus, it takes 57.1 hours (two days and about nine hours) until the amount of radioactive iodine is below one tenth of a milligram.

7. Glyphosate is an ingredient in many herbicides. When it lands on a plant it is quickly absorbed and broken down, but when it lands on soil it breaks down slowly and kills beneficial microbes in the process. The average half-life in soil is 40 days (adapted from Landkamp/Hull).

- Determine the daily rate of decay.
- If 4 kg are used on one hectare, with 1 kg landing on soil, then how long will it take until only 100 g are left?

Solution: We measure the glyphosate content in gram and time in days. Then the amount in the soil after  $x$  days is (1kg = 1,000g)

$$y = 1,000 \cdot 2^{-x/40}$$

- The multiplier is  $M = 2^{-1/40}$ , because

$$\left(2^{-1/40}\right)^x = 2^{-\frac{1}{40}x} = 2^{-x/40}$$

and therefore the rate is

$$\begin{aligned} r &= M - 1 = 2^{-1/40} - 1 \\ &= 2^{-0.025} - 1 = -0.0172 \end{aligned}$$

and we conclude that the substance decays at a rate of 1.7% per day.

- We begin with an estimate: Originally we have 1,000g. After 40 days there are 500g in the soil, after 80 days the amount will be reduced to 250g, after 120 days we have 125g left, but after 160 days we are down to 62.5g. The result should be near 140 days.

And now for the analytical solution: We set  $y = 100$  and solve for  $x$ :

$$\begin{aligned} 100 &= 1,000 \cdot 2^{-x/40} \\ 2^{-x/40} &= \frac{100}{1,000} = \frac{1}{10} \\ 2^{x/40} &= 10 \\ 2^x &= 10^{40} \\ x &= \frac{\log(10^{40})}{\log 2} = \frac{40}{0.301} \\ &= 132.9 \end{aligned}$$

In the third step we took reciprocals on both sides (get rid of minus signs), and in the following step we raised both sides to the 40th power (get rid of the fraction). As a result we see that it takes about 133 days (4.4 months) until only 100g are left.

Check the result: When we substitute  $x = 132.9$  into the model, we find that  $1,000 \cdot 2^{-132.9/40} = 100$ .

A graph of  $y = 1,000 \cdot 2^{-x/40}$  reveals that  $y = 100$  for  $x \approx 133$ , which confirms our result as well.



8. The population of the United States since 1950 is given in the table below

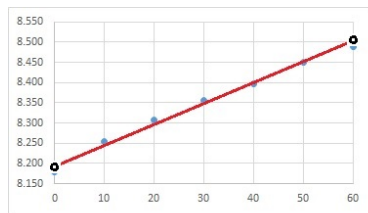
Year	Population
1950	151,325,798
1960	179,323,175
1970	203,211,926
1980	226,545,805
1990	248,709,873
2000	281,421,906
2010	308,745,538

Use an exponential model and the straight-edge method to approximate the annual growth rate.

Solution: We use the variable  $x$  to count the years since 1950 and  $y$  for the population data. Taking the logarithms results in the table below.

$x$	$\log y$
0	8.180
10	8.254
20	8.308
30	8.355
40	8.396
50	8.449
60	8.490

In the next step we graph this information and find an approximating line.



Using the points  $(0, 8.19)$  and  $(60, 8.5)$ , we see that the slope is

$$m = \frac{8.5 - 8.19}{60 - 0} = \frac{0.31}{60} = 0.00517$$

We do not need the full equation of the line. Recall that

$$y = y_0 M^x \\ \Leftrightarrow \log y = x \log M + \log y_0$$

and therefore

$$\log M = m = 0.00517$$

This implies that

$$M = 10^{0.00517} = 1.01197$$

and we see that the annual growth rate of the U.S. for that period was

$$r = 0.01197 = 1.2\%$$

Least squares regression by computer yields a growth rate of 1.167%.

9. Construct the inverse function of

$$f(x) = y_0 M^x$$

Solution: Recall that in order to find the inverse function, we have to solve the equation  $y = f(x)$  for  $x$ . In our case we obtain

$$y = f(x) = y_0 M^x \\ M^x = \frac{y}{y_0} \\ x = \frac{\log \frac{y}{y_0}}{\log M}$$

where in the last step we used the formula for exponential equations (14). After renaming the variables we obtain

$$g(x) = \frac{\log \frac{x}{y_0}}{\log M}$$

as the inverse function of  $f$ .

In the very special case where  $y_0 = 1$  and  $M = 10$ , we see that

$$g(x) = \frac{\log x}{\log 10} = \log x$$

is the inverse function of  $f(x) = 10^x$ .

## 14.7 Exercises

- Assume that  $y$  changes exponentially with  $x$ . Complete the tables below and find an equations of the form  $y = y_0 M^x$ .

(a)

$x$	$y$
0	6
1	
2	
3	750
4	

(b)

$x$	$y$
0	500
1	
2	125
3	
4	

(c)

$x$	$y$
0	
1	4
2	
3	5
4	

- Sketch the graph of the functions  $y = 5 \cdot 2^x$  and  $y = 250 \cdot 0.8^x$  on a logarithmic scale.
- Express  $y = 5 \cdot 2^{-x/4}$  in the form  $y = y_0 M^x$ . What are  $y_0$ ,  $M$  and  $r$ ?
- The population of the City of Radford grew from 15,859 in 2000 to 16,408 in 2010. Assuming an exponential model, what is the annual growth rate?
- A population of 200 deer grows at an annual rate of 8%.
  - What is the population after 5 years?
  - What is the doubling time?
- A substance grows according to the law  $y = 15 \cdot 1.05^x$ . What is the doubling time?
- A bacteria culture doubles in size every 45 minutes.
  - What is the hourly growth rate?
  - Initially 250 bacteria are present. How long will it take until the colony has one million bacteria?
- A bacteria culture grows from 600 bacteria to 13,000 bacteria in two hours. What is the generation time (the time it takes to double in size)?
- Scientists discovered a new radioactive substance which decayed by 20% within one hour. What is its half-life?
- A pollutant has a half-life of 4 months.
  - What is the monthly rate of decay?
  - What is the annual decay rate.
  - If initially 600 g of the substance are present, how long will it take until the amount is reduced to 1 g?
- The half life of the radioactive isotope  $^{14}\text{C}$  is 5730 years. A piece of wood measures only 20% of the common  $^{14}\text{C}$ : $^{12}\text{C}$  ratio. How old is it?
- A pain killer has a half life of eight hours, i.e. the body absorbs one half of the drug within eight hours.
  - A patient is given a 12mg dose. How much is left in the body after one day?
  - What dosage is required so that after six hours there are still 6 mg of the drug left in the body?

## Answers

1. (a)  $y = 6 \cdot 5^x$

$x$	$y$
0	6
1	30
2	150
3	750
4	3750

(b)  $y = 500 \cdot 0.5^x = 500 \cdot 2^{-x}$

$x$	$y$
0	500
1	250
2	125
3	62.5
4	31.25

(c)  $y = 3.578 \cdot 1.118^x = \sqrt{12.8} \cdot 1.25^{x/2}$

$x$	$y$
0	3.578
1	4
2	4.472
3	5
4	5.590

2.

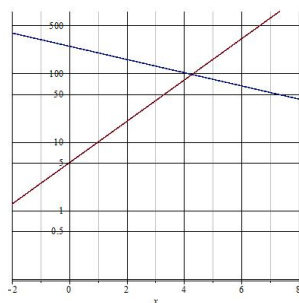


Figure 32: Problem 2

3.  $y = 5 \cdot \left(2^{-1/4}\right)^x = 5 \cdot 0.8409^x$   
 $y_0 = 5$ ,  $M = 0.8409$ ,  
 $r = -0.1591 = -15.91\%$

4.  $r = 0.3409\%$

5. (a) 294 (293.8656)

(b) 9 years (9.0065)

6. 14.2 (14.207)

7. (a) 152%

(b) 9 hours (8.974)

8. 27 minutes

9. 3.1 hours (3.1063)

10. (a) 15.9% ( $r = -0.159$ )

(b) 87.5% ( $r = -0.875$ )

(c) 3 years (3.076)

11. 3,300 years (13,304.6)

12. (a) 1.5 mg

(b) 10 mg (10.091)

## 15 Power Functions

Power functions are very popular with biologists, in particular in the study of allometry. Kleiber's Law, which relates body mass and basal metabolic rates of mammals, is probably the most celebrated example. Species-area curves (the number of species in a given area) are another application of power functions.

### 15.1 Definition

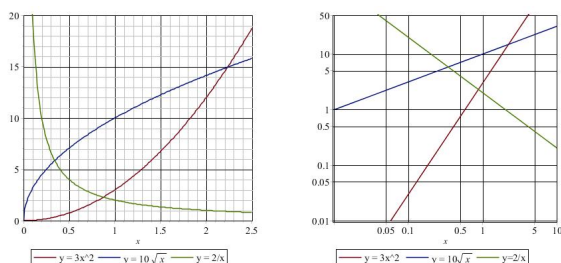
**Definition:** Power functions have the form

$$y = c x^a \quad (21)$$

Examples of power functions include

$$\begin{aligned} y &= 3x^2 \\ y &= 10 \sqrt{x} = 10x^{1/2} \\ y &= \frac{2}{x} = 2x^{-1} \end{aligned}$$

In a way, it is much easier to work with power functions than with polynomials, because they only have one term. However, there is no restriction on the power  $a$ . Any number is permitted, not just positive integers.



At first glance, there is nothing special about the graphs of power functions, but we obtain lines when we plot power functions on a loglog scale (logarithms on both axes).

The reason is quite simple: Take the logarithm of  $y$  and simplify; then

$$\begin{aligned} \log y &= \log(c x^a) = \log c + \log x^a \\ &= a \log x + \log c \end{aligned}$$

If we introduce the new variables

$$u = \log x \quad \text{and} \quad v = \log y$$

we have the equation

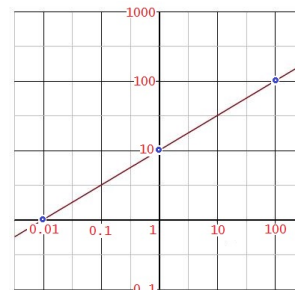
$$v = ax + \log c$$

which is an equation of a line with slope  $a$  and  $v$ -intercept  $\log c$ .

**Example:** Consider the power function  $y = 10\sqrt{x}$ . Then

$$\begin{aligned} v &= \log y = \log(10\sqrt{x}) \\ &= \log 10 + \log x^{1/2} = 1 + \frac{1}{2} \log x \\ &= \frac{1}{2} u + 1 \end{aligned}$$

In a loglog plot we graph the line  $v = \frac{1}{2}u + 1$ , and "cheat" when we label the axes.



This graph still represents  $y = 10\sqrt{x}$ . The points for

$$\begin{aligned} x = 1 & \quad y = 10\sqrt{1} = 10 \\ x = 100 & \quad y = 10\sqrt{100} = 100 \\ x = 0.01 & \quad y = 10\sqrt{0.01} = 1 \end{aligned}$$

are highlighted on the graph.

Matching a power function to data is a little tricky. We begin with an example.

$x$	$y$
2	20
5	125

Because  $y = cx^a$ , this table translates into the set of equations

$$\begin{aligned} 20 &= c 2^a \\ 125 &= c 5^a \end{aligned}$$

with  $a$  and  $c$  as unknowns. Division yields

$$\begin{aligned} \frac{125}{20} &= \frac{c 5^a}{c 2^a} \\ 6.25 &= \frac{5^a}{2^a} = \left(\frac{5}{2}\right)^a = 2.5^a \end{aligned}$$

Unless you realize that  $2.5^2 = 6.25$ , you need logarithms to solve this exponential equation, and obtain

$$a = \frac{\log 6.25}{\log 2.5} = 2$$

Once  $a = 2$  has been identified, we substitute this value into the first equation and solve for  $c$ :

$$20 = c 2^2 = 4c \Rightarrow c = 5$$

and the underlying power function becomes

$$y = 5x^2$$

Usually the exponent  $a$  is the more relevant part of a power function, and if data  $(x_1, y_1)$  and  $(x_2, y_2)$  are known, the exponent  $a$  can be computed from

$$a = \frac{\log y_2/y_1}{\log x_2/x_1} \quad (22)$$

The verification of this result is a routine exercise with logarithms. For our last example we would calculate

$$a = \frac{\log 125/20}{\log 5/2} = \frac{\log 6.25}{\log 2.5} = 2$$

in order to find the exponent.

Check the worked problems for more examples and applications.

## 15.2 Approximation with Power Functions

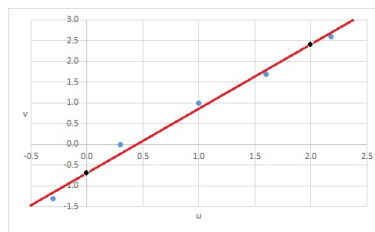
Data are never perfect, and approximations are needed. We can adapt the straight-edge method to power functions and use the following steps:

1. Take the logarithms of the  $x$ - and  $y$ -values of the data.
2. Carefully plot the logarithmic data.
3. Find an approximating line by the straight-edge method (see Section 12.3).
4. Convert the approximating line back into a power function.

**Example:** The data and the respective logarithms are given in the table.

$x$	$y$	$u = \log x$	$v = \log y$
0.5	0.05	-0.301	-1.301
2	1	0.301	0
10	10	1	1
40	50	1.602	1.699
150	400	2.176	2.602

The graph below shows the logarithms in the  $uv$ -plane. The points  $(u, v) = (0, -0.7)$  and  $(u, v) = (2, 2.4)$  were selected for the approximating line.



In the final step we have to convert the line to a power function with the original variables  $x$  and  $y$ . Since  $u = \log x$  and  $v = \log y$ , it follows that

$$x = 10^u \quad \text{and} \quad y = 10^v$$

There are two strategies to complete Step 4:

Option One: Convert the points back to the  $xy$ -system and construct the power function.

$$\begin{array}{c|c} u & v \\ \hline 0 & -0.7 \\ 2 & 2.4 \end{array} \Rightarrow \begin{array}{c|c} x & y \\ \hline 1 & 0.2 \\ 100 & 251 \end{array}$$

We get  $c = 0.2$  directly, because the value for  $x = 1$  is known. For the parameter  $a$  we use formula (22) and find that

$$a = \frac{\log 251/0.2}{\log 100/1} = \frac{3.1}{2} = 1.55$$

and the interpolating power function is

$$y = 0.2 \cdot x^{1.55}$$

Option Two: Determine the equation for the interpolating line in the  $uv$ -plane, and convert it to the original variables.

The  $v$ -intercept  $(0, -0.7)$  is a given point, and the slope becomes

$$m = \frac{2.4 - (-0.7)}{2 - 0} = \frac{3.1}{2} = 1.55$$

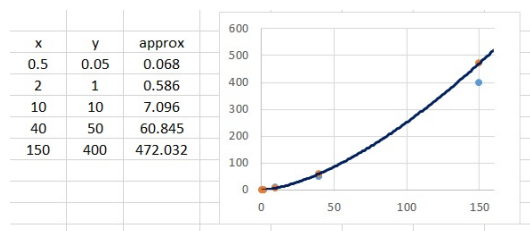
Therefore the line in the  $uv$ -plane is

$$v = 1.55u - 0.7$$

It now follows that

$$\begin{aligned} y &= 10^v = 10^{1.55u - 0.7} \\ &= 10^{-0.7} \cdot (10^u)^{1.55} = 0.2 \cdot x^{1.55} \end{aligned}$$

Of course either option yields the same approximation. A graph of the points along with the power approximation is shown below.



Graphing calculators or computer software have built-in functions for this task, based upon the least squares approximation principle.

On a graphing calculator enter the points in lists with  $x$ - and  $y$ -coordinates (STAT menu and Edit option). Once this is completed, use the STAT menu, followed by CALC and select the PwrReg option. For the data from our example, the calculator should respond with  $y = a * x \wedge b$  where  $a = 0.223$  and  $b = 1.52$ .

In EXCEL one should graph the points in a scatter plot first and then select "Power" for the trendline option. If you ask to display the equation on the chart, the formula  $y = 0.223x^{1.52}$  will appear.

By the way, our straight-edge approximation  $y = 0.2x^{1.55}$  is fairly close to the least squares approximation of the computing devices.

### 15.3 Worked Problems

- Find a power function  $y = c x^a$  for the data below.

(a) 
$$\begin{array}{c|c} x & y \\ \hline 1 & 3 \\ 2 & 48 \end{array}$$

(b) 
$$\begin{array}{c|c} x & y \\ \hline 1 & 3 \\ 2 & 30 \end{array}$$

(c) 
$$\begin{array}{c|c} x & y \\ \hline 2 & 6 \\ 5 & 20 \end{array}$$

$$(d) \quad \begin{array}{c|c} x & y \\ \hline 10 & 1,000 \\ 20 & 25 \end{array}$$

Solutions:

If we substitute  $x = 1$  into the formula for a power function (21) we always obtain  $y = c \cdot 1^a = c$ , regardless of the value of  $a$ . We will take advantage of this shortcut in problems (a) and (b).

In part (b) we will show detailed steps to find  $a$  and  $c$ , otherwise we will apply formula (22) for brevity.

(a) Formula (22) implies that

$$\begin{aligned} a &= \frac{\log y_2/y_1}{\log x_2/x_1} = \frac{\log 48/3}{\log 2/1} \\ &= \frac{\log 16}{\log 2} = 4 \end{aligned}$$

$c = 3$  because we have information at  $x = 1$ , and the power function becomes  $y = 3x^4$ .

(b) The given information translates into

$$\begin{aligned} 3 &= c \cdot 1^a \\ 30 &= c \cdot 2^a \end{aligned}$$

The first equation becomes  $c = 3$ , and after substitution we see that

$$2^a = \frac{30}{3} = 10$$

The solution of this exponential equation is

$$a = \frac{\log 10}{\log 2} = \frac{1}{0.301} = 3.322$$

and the power function is  $y = 3x^{3.322}$ .

(c) Here we obtain from (22)

$$\begin{aligned} a &= \frac{\log y_2/y_1}{\log x_2/x_1} = \frac{\log 20/6}{\log 5/2} \\ &= \frac{\log 3.333}{\log 2.5} = 1.314 \end{aligned}$$

We use the first point and substitution in order to find  $c$

$$\begin{aligned} 6 &= c \cdot 2^{1.314} = 2.486 \cdot c \\ c &= \frac{6}{2.486} = 2.413 \end{aligned}$$

and thus  $y = 2.413 \cdot x^{1.314}$ .

(d) In this exercise we see notice that the  $y$ -data are decreasing, which forces a negative exponent. From (22) we find that

$$\begin{aligned} a &= \frac{\log y_2/y_1}{\log x_2/x_1} = \frac{\log 25/1,000}{\log 20/10} \\ &= \frac{\log 0.025}{\log 2} = -5.322 \end{aligned}$$

Substitution with the first point yields

$$\begin{aligned} 1,000 &= c \cdot 10^{-5.322} \\ c &= \frac{1,000}{10^{-5.322}} = \frac{10^3}{10^{-5.322}} \\ &= 10^{3+5.322} = 10^{8.322} \cdot 10^8 \\ &= 2.099 \cdot 10^8 \end{aligned}$$

and the underlying power function is

$$y = 2.099 \cdot 10^8 \cdot x^{-5.322}$$

This function can be manipulated into  $y = 1,000 \cdot \left(\frac{10}{x}\right)^{5.322}$ . Try it.

2. Interpolate the data

$x$	5	16	100	300	1,500
$y$	75	50	20	5	2.5

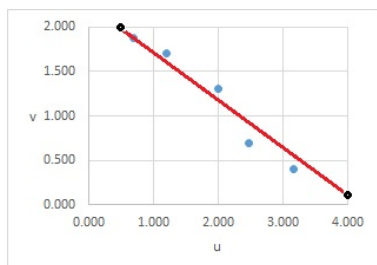


by a power function. Use the straight-edge method and compare this estimate to the least squares fit from a computer or calculator.

Solution: The first step calls for taking the logarithm of the data.

$u = \log x$	$v = \log y$
0.699	1.875
1.204	1.699
2.000	1.301
2.477	0.699
3.176	0.398

A graph in the  $uv$ -plane (Step 2) and an approximating line (Step 3) are shown in the figure.



The line passes through the points  $(0.5, 2)$  and  $(4, 0.1)$ . The slope of this line is

$$m = \frac{0.1 - 2}{4 - 0.5} = \frac{-1.9}{3.5} = -0.543$$

and its equation becomes (see Point-Slope Form (5))

$$\begin{aligned} v - 2 &= -0.543(u - 0.5) \\ v &= -0.543u + 2.271 \end{aligned}$$

Now we convert to  $xy$ -variables:

$$\begin{aligned} y &= 10^v = 10^{-0.543u + 2.271} \\ &= 10^{2.271} \cdot (10^u)^{-0.543} \\ &= 187 x^{-0.543} \end{aligned}$$

This was Option Two. The steps for Option One require to convert the points to  $xy$ -coordinates first, the result being  $(3.16, 100)$  and  $(10,000, 1.26)$ , and then to construct the interpolating power function. Answers will be identical for either option.

The least squares method yields the approximation

$$y = 251.67 x^{-0.631}$$

3. **Basal Metabolic Rates.** This is a classical problem for power functions. The data go back to research done by Max Kleiber in the 1930ies who compared body mass of mammals and the respective basal metabolic rates (oxygen consumption at rest). Some of the data are listed below. The mass is measured in gram, the metabolic rate in milliliter (mL) of oxygen per hour.

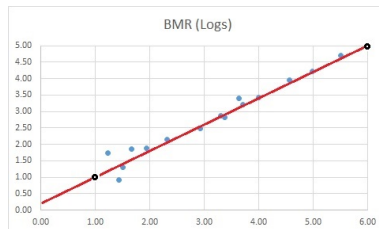
	Mass	BMR
Moose	325000	51419
Lion	98000	16954
Cougar	37200	8842
Coyote	10000	2687
Raccoon	5075	1599
Red fox	4400	2442
Rock hyrax	2400	660
Possum	2005	731.6
Meerkat	850	310
Ground Squirrel	205.4	140.4
Indian Gerbil	87	75.7
Chipmunk	45.8	72.7
Naked mole rat	32	20.5
Bat	27	8.3
Shrew	17.1	54.7

Problem: Use the straight-edge method to find an approximating power function with  $x$  representing body mass and  $y$  representing oxygen consumption.

Solution: First we have to calculate the logarithms of the data.

	u	v
Moose	5.51	4.71
Lion	4.99	4.23
Cougar	4.57	3.95
Coyote	4.00	3.43
Raccoon	3.71	3.20
Red fox	3.64	3.39
Rock hyrax	3.38	2.82
Possum	3.30	2.86
Meerkat	2.93	2.49
Ground Squirrel	2.31	2.15
Indian Gerbil	1.94	1.88
Chipmunk	1.66	1.86
Naked mole rat	1.51	1.31
Bat	1.43	0.92
Shrew	1.23	1.74

The next task is to graph these points and to eyeball an interpolating line.



For simplicity we use the line containing the points (1, 1) and (6, 5) in the  $uv$ -plane. We choose Option One to complete the problem, and we convert the points back to the  $xy$ -plane. Using  $x = 10^u$  and  $y = 10^v$ , we find the points

$x$	$y$
10	10
$10^6$	$10^5$

Now we use formula (22) to construct the

exponent:

$$a = \frac{\log \frac{10^5}{10}}{\log \frac{10^6}{10}} = 0.8$$

Then we substitute the first point and  $a = 0.8$  into the equation  $y = c \cdot x^a$  and obtain

$$10 = c \cdot 10^{0.8}$$

which implies that

$$c = \frac{10}{10^{0.8}} = 10^{0.2} = 1.6$$

and our approximation is

$$y = 1.6 x^{0.8}$$

With Option Two we would use the equation  $v = 0.8u + 0.2$  and take it from there.

The least squares result (computer/calculator) for these data yields

$$y = 1.9726 x^{0.7918}$$

## 15.4 Exercises

- Sketch the function  $y = 4\sqrt{x}$  on a loglog scale. Try these options:
  - Sketch your own loglog graphing paper (1, 10, 100, ... equally spaced on both axes) and graph the function.
  - Compute  $u = \log x$  and  $v = \log y$  for selected points and graph  $u$  versus  $v$ .
- Assume that  $y = cx^a$  is a power function. Find  $c$  and  $a$  and complete the tables.

	$x$	$y$
(a)	1	0.1
	2	
	5	
	10	10

(b)

$x$	$y$
1	3
3	1
6	

(c)

$x$	$y$
1	3
4	
8	12

3. Sketch the functions

(a)  $y = 0.01x^3$

(b)  $y = \frac{10}{x^2}$

(c)  $y = 2.3x^{0.7}$

on a regular grid and on a loglog grid.

4. Let  $y = 4x^{1.5}$ .

(a) Find  $x$  such that  $y = 500$ .

(b) Find  $x$  such that  $y = 100$ .

5. Find a power function approximation for the data in the table. Use the straight-edge method for power functions and compare the approximations to the given data. In addition, use a calculator and compare the calculator results to your straight-edge results.

(a)

$x$	1	2	5	10
$y$	2	20	77	250

(b)

$x$	$y$
0.2	1.1
4	40
10	120
25	360
60	1,000
100	1,900

(c)

$x$	$y$
1	425
4	250
15	140
60	80
250	48
1,000	25

6. You investigate how the diversity of grasses on a meadow changes with area. On the smallest plot of 50 cm  $\times$  50 cm you find just two species. As you increase the lot size you detect more and more species. On the largest plot (50 m  $\times$  50 m) you count 31 different species. The data are summarized in the table below.

Lot size( $m^2$ )	No. of Species
0.25	2
1	4
12	7
50	11
120	13
600	21
2,500	31

Use the straight edge method to find an approximating power function with  $x$  representing area, and  $y$  for the number of species.

7. The table below contains the heart rate at rest for selected mammals (Kong, NIU).

	Mass (g)	Pulse
Mouse	25	670
Rat	200	420
Guinea pig	300	300
Rabbit	2,000	205
Small dog	5,000	120
Large dog	30,000	85
Man	70,000	72
Horse	450,000	38

Find an approximating power function by the straight edge method. Use  $x$  for the mass and  $y$  for the heart rate.

## Answers

1.

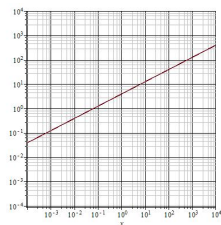


Figure 33: Problem 1 (a)

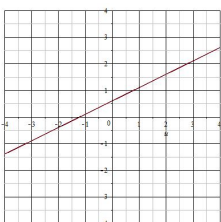


Figure 34: Problem 1 (b)

2. (a)  $y = 0.1x^2$

$x$	$y$
1	0.1
2	0.4
5	2.5
10	10

(b)  $y = \frac{3}{x}$

$x$	$y$
1	3
3	1
6	0.5

(c)  $y = 3x^{2/3}$

$x$	$y$
1	3
4	$7.56 = 6\sqrt[3]{2}$
8	12

3.

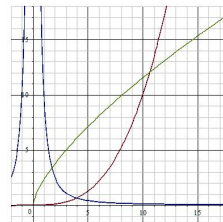


Figure 35: Problem 3

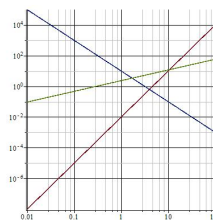


Figure 36: Problem 3

4. (a)  $x = 25$

(b)  $x = 25^{2/3} = 8.550$

5. Answers may vary.

(a)  $y \approx 2.5x^2$ , use  $(u, v) = (0, 0.4)$  and  $(u, v) = (1, 2.4)$

$x$	1	2	5	10
$y$	2	20	77	250
est.	2.5	10	62.5	250

Calculator:  $y = 2.92x^{2.01}$

(b)  $y = 7x^{1.2}$ ,  
use  $(u, v) = (-0.7, 0)$  and  
 $(u, v) = (2, 3.3)$  and some rounding.

x	y	est.
0.2	1.1	1.015
4	40	36.95
10	120	110.9
25	360	333.1
60	1000	952.5
100	1900	1758

Calculator:  $y = 7.581x^{1.197}$

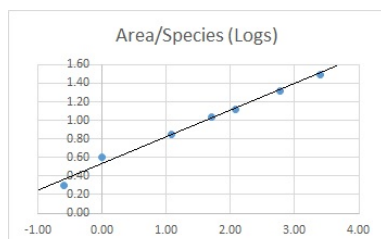
- (c)  $y = 400x^{-0.4}$ ,  
 use  $(u, v) = (0, 2.6)$  and  
 $(u, v) = (3, 1.4)$

x	y	est.
1	425	400.0
4	250	229.7
15	140	135.4
60	80	77.77
250	48	43.94
1000	25	25.24

Calculator:  $y = 429.8x^{-0.407}$

6. Answer may vary.

$$y = 3.4 x^{0.3}$$



7. Answer may vary.

$$y = 2,000 x^{-0.3}$$



## 16 Difference Equations

Difference equations are a powerful modeling tool, which can be easily investigated with computer simulations. Population growth, predator-prey, host-parasite or disease modeling are typical applications.

This section makes heavy use of the function notation, and a review of Chapter 11 is advisable.

### 16.1 Introduction

In this introductory section we study an example which illustrates the main ideas and concepts behind difference equations.

The Problem: We monitor the deer population in a large park, which currently consists of 1,200 deer. We know that the population increases at 10% annually, and hunting reduces the herd by 100 animals each year. Predict the population in the future.

This looks a bit like the discussion of the the monkey population in the beginning of Chapter 14. We start with a population of 1,200. The 10% increase results in 120 more deer in the following year, but we also have to subtract the 100 deer killed by hunters. The new population becomes

$$1,200 + 120 - 100 = 1,220$$

In the next year we go through the same process, starting with 1,220 deer, and we obtain

$$1,200 + 122 - 100 = 1,242$$

Repeating the steps one more time leads to<sup>31</sup>

$$1,242 + 124.2 - 100 = 1,266.2$$

<sup>31</sup>Round to the nearest integer if the decimal bothers you.

The results are summarized in the table below.

Year	Population
0	1,200
1	1,220
2	1,242
3	1,266.2

Our next step is to derive a notation so that we can express our model mathematically. We let  $n$  be the number of years from the original count, and we use  $p(n)$  for the population<sup>32</sup> in year  $n$ .

With this notation our table is to be interpreted as

Year $n$	Population $p(n)$
$n = 0$	$1,200 = p(0)$
$n = 1$	$1,220 = p(1)$
$n = 2$	$1,242 = p(2)$
$n = 3$	$1,266.2 = p(3)$

$p(2)$  means the population in the second year, and  $p(10)$  stands for the population ten years later, etc.

What can we say about our model? We know that  $p(0) = 1,200$ , since this is the original deer count, but we don't have an explicit formula for the deer population in following years. Instead, we have a recipe how to move from one year to the next. If we want the deer population in the year  $n$ , which is written as  $p(n)$ , we have to add 10% to last year's population, denoted by  $p(n-1)$ , and subtract 100. This results in the equation

$$p(n) = p(n-1) + 0.1 p(n-1) - 100$$

which shortens to

$$p(n) = 1.1 p(n-1) - 100 \quad (23)$$

---

<sup>32</sup>We use the function notation.  $p(n)$  is pronounced as "p of n", and in this example it means the population in year  $n$ .

This is an example of a difference equation. We need a starting value, and everything else follows by a domino effect. When  $n = 1$ , the equation (23) becomes

$$\begin{aligned} p(1) &= 1.1 p(0) - 100 \\ &= 1.1 \cdot 1,200 - 100 = 1,220 \end{aligned}$$

Substituting  $n = 2$  into (23) leads to

$$\begin{aligned} p(2) &= 1.1 p(1) - 100 \\ &= 1.1 \cdot 1,220 - 100 = 1,242 \end{aligned}$$

and when  $n = 3$  the difference equation (23) becomes

$$\begin{aligned} p(3) &= 1.1 p(2) - 100 \\ &= 1.1 \cdot 1,242 - 100 = 1,266.2 \end{aligned}$$

and so on.

It is a unique feature of difference equations that we can determine  $p(n)$  only for integers.  $p(2) = 1,242$  and  $p(3) = 1,266.2$ , but  $p(2.4)$  is unspecified.

This is in stark contrast to exponential models. If we set hunting aside, the difference equation becomes

$$p(n) = 1.1 p(n-1)$$

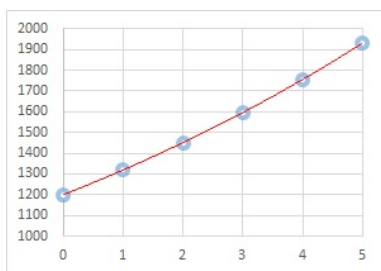
In an equivalent exponential model we would use  $r = 0.1$ ,  $M = 1 + r = 1.1$  and  $y_0 = 1,200$ , with solution

$$y = 1,200 \cdot 1.1^x \quad (24)$$

It can be shown that  $p(n)$  matches the exponential growth formula<sup>33</sup> (24), but it is only defined for integers. On the other hand, there is no problem with setting  $x = 2.4$  in (24) to obtain  $y = 1,200 \cdot 1.1^{2.4} = 1,508.4$ . Graphically, the exponential model leads to a smooth curve (red), while the difference equation produces a scatter plot (blue).

---

<sup>33</sup> $p(n) = 1,200 \cdot 1.1^n$



This graph also contrasts the difference between discrete and continuous variables. The  $p(n)$  are defined at the positive integers only, which makes  $n$  a discrete variable, while  $y$  is defined for all  $x$ , and  $x$  is a continuous variable.

It is usually very difficult and not very useful to find explicit solutions to difference equations. Instead, we let computers do numerical calculations, and focus on qualitative features, such as

- will the points grow indefinitely?
  - will the population become extinct?
  - will the system settle for a steady state?
- and so on.

## 16.2 First Order Difference Equations

We will use the letter  $p$  to denote the *state variable*, and the state of the system at a particular time  $n$  is denoted by  $p(n)$ . In difference equations, the state of the system changes as time progresses. This change is driven by an *update function*  $f$ , which depends on the state of the system itself.

In its most general form, a first order difference equation has the structure

$$p(n) = p(n-1) + f(p(n-1)) \quad (25)$$

where

- $n \geq 0$  *counting index*, must be an integer, usually time (years, days, seconds or even generations)
- $p(n)$  *state of the system* at time  $n$ , frequently a population
- $p(0)$  *initial condition*
- $f(p)$  *update function*, describes how  $p$  changes

A difference equation has the characteristics of a feed-back loop: For the given state  $p(n-1)$ , we identify the update  $f(p(n-1))$ , and then change the state of the system in the next time step to  $p(n) = p(n-1) + f(p(n-1))$ .

The update function  $f(p)$ , along with the initial condition  $p(0)$ , determine the behavior of the system in the future. When  $p(0)$  is given, we can use  $n = 1$  in (25) to obtain

$$p(1) = p(0) + f(p(0))$$

which makes  $p(1)$  a known quantity. Now set  $n = 2$ , then

$$p(2) = p(1) + f(p(1))$$

is known, and continuing in this fashion, we can compute  $p(n)$  for any number  $n$  (we just have to be patient if  $n$  is very big, or use a computer to speed things up).

After this abstract and fairly general introduction, and we turn to specific examples.

**Example:** The opening example we had 1,200 deer to begin with. This makes for

$$p(0) = 1,200$$

as initial condition. The change from one year to the next was the 10% population increase minus the deer lost to hunting. Therefore the update function becomes

$$f(p) = 0.1p - 100$$

and we can now assemble the difference equation according to (25):

$$p(n) = p(n-1) + 0.1p(n-1) - 100$$

which is equivalent to (23).

Let's take another look at the  $f(p)$  term. Initially we have  $p = 1,200$  deer, and therefore (substitute)

$$f(1,200) = 0.1 \cdot 1,200 - 100 = 20$$

Thus, the population will increase by 20 animals, and the new population becomes  $p = 1,220$ . When we substitute this number into  $f$ , we find that

$$f(1,220) = 0.1 \cdot 1,220 - 100 = 22$$

and we will have  $1,220 + 22 = 1,242$  deer in the next year. If for some reason the deer population becomes 2,000, then

$$f(2,000) = 0.1 \cdot 2,000 - 100 = 100$$

and the following year will see 2,100 deer.

Slogan: If you want to calculate the new value  $p(n)$ , plug the current value  $p(n-1)$  into the formula for  $f$ , and add the result to  $p(n-1)$ .

**Example:** Research suggests that a migratory bird population decreases by 8% annually. The last count in 2010 showed 7,600 birds.

We construct a difference equation for this scenario: If  $n$  stands for the years after 2010, we have  $p(0) = 7,600$  as initial condition. The 8% decrease can be modeled by

$$f(p) = 0.92p$$

and the difference equation becomes

$$\begin{aligned} p(n) &= p(n-1) - 0.08p(n-1) \\ &= 0.92p(n-1) \end{aligned}$$

**Example:** Consider

$$\begin{aligned} a(n) &= a(n-1) + 12 \\ a(0) &= 100 \end{aligned}$$

Here  $a$  increases by 12 each time ( $f(a) = 12$  is a constant function), beginning with  $a(0) = 100$ . The next terms are

$$\begin{aligned} a(1) &= a(0) + 12 = 100 + 12 = 112 \\ a(2) &= a(1) + 12 = 112 + 12 = 124 \\ a(3) &= a(2) + 12 = 124 + 12 = 136 \end{aligned}$$

and so on.

**Example (New Car Loan):** This example is not biological, but the same ideas apply: You borrow \$10,000 for a new car. The interest rate is 4.8%, and you make monthly payments of \$400.

Let  $B(n)$  stand for the balance on your loan after  $n$  months. Clearly,  $B(0) = 10,000$ , which is the original loan amount. From one month to the next, your debt increases by  $\frac{4.8\%}{12} = 0.4\%$ , while your payment lowers the debt by \$400. The balance  $B$  on your loan will change by

$$f(B) = 0.004B - 400$$

and the difference equation becomes

$$\begin{aligned} B(n) &= B(n-1) + 0.004B(n-1) - 400 \\ &= 1.004B(n-1) - 400 \end{aligned}$$

After one month your outstanding debt is

$$\begin{aligned} B(1) &= 10,000 + 0.004 \cdot 10,000 - 400 \\ &= 10,000 + 40 - 400 = 9,640 \end{aligned}$$

In words: You owe \$10,000, you accumulate \$40 in interest, and your payment reduces the debt by \$400.

It is very tedious (and dull) to continue these computations by hand. The resulting



data are known as an amortization schedule in the business world. As it turns out, the loan will be paid off in just 27 months. In the real world, customers take out four year loans, and the monthly payments in this case would be \$229.39, rather than \$400. This is done by adjusting the payments so that  $B(48) \approx 0$ .

As the models get more sophisticated, the update function  $f$  will be more complex, and the difference equation may become very cluttered. When you experiment with difference equations and you calculate iterates without a computer, it may be advisable to break up the steps and compute the update  $f(p(n-1))$  first, and then form  $p(n) = p(n-1) + f(p(n-1))$ .

**Example:** We take  $p(0) = 25$  as initial population and use the update function

$$f(p) = \frac{p \cdot (100 - p)}{p + 100}$$

This is a special case of the Beverton-Holt model, which in turn is a variation of logistic growth, which we will study below. The resulting equation becomes

$$\begin{aligned} p(n) \\ = p(n-1) + \frac{p(n-1)(100 - p(n-1))}{p(n-1) + 100} \end{aligned}$$

is not terribly confusing, nonetheless, we will compute the update first, and then form  $p(n)$ .

We have  $p(0) = 25$ . Now we form  $f(25)$ , that is, we substitute 25 for  $p$ , and obtain

$$f(25) = \frac{25 \cdot (100 - 25)}{25 + 100} = \frac{25 \cdot 75}{125} = 15$$

This is the update, and the new population is  $p(1) = 25 + 15 = 40$ .

Now we repeat the process with  $p = 40$ . Then

$$f(40) = \frac{40 \cdot (100 - 40)}{40 + 100} = \frac{120}{7} = 17.14$$

and the new population is  $p(2) = 40 + 17.14 = 57.14$ .

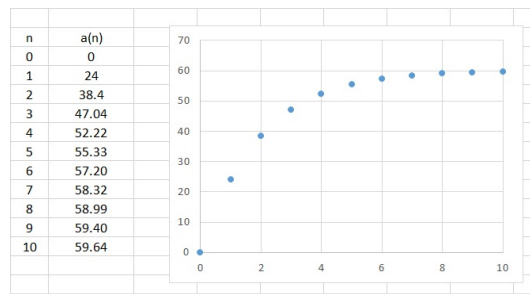
At this point the computations become messy. We summarize our results for the first five iterations in a table.

$n$	$p(n)$	$f(p(n))$
0	25	15
1	40	17.14
2	57.14	15.58
3	72.73	11.48
4	84.21	7.22
5	91.43	

**Graphs.** As mentioned in the introduction, scatter plots are a natural way to depict the results of difference equations. The inputs are integers, and the gaps between the points are a characteristic feature. The graph below shows the solution to

$$a(n) = 0.6a(n-1) + 24$$

$$a(0) = 0$$



**Terminology and Notation.** A difference equation is usually written as

$$p(n) - p(n-1) = f(p(n-1))$$

which emphasizes the change of the state  $p$ . If we set

$$\Delta p = p(n) - p(n-1)$$

it becomes

$$\Delta p = f(p(n-1))$$

and we see that the change of  $p$  is determined by the update function  $f$ . This approach allows for a nice connection to the "differential equation"

$$\frac{dp}{dx} = f(p)$$

which the continuous variable counterpart to a difference equation. Differential equations are a Calculus based field of mathematics, and they are widely used in science and engineering, and they are becoming an increasingly important tool in the life sciences.

Equations of the form

$$p(n) = g(p(n-1))$$

are called *recurrence relations* or *iterative equations*, with  $g$  being the iteration function. The two concepts are completely equivalent when we use

$$g(p) = p + f(p)$$

but the perspective is different. For example, the difference equation

$$p(n) - p(n-1) = 0.2p(n-1)$$

emphasizes that  $p$  increases by  $0.2 = 20\%$  each time, while the same process written as

$$p(n) = 1.2p(n-1)$$

indicates that  $p$  grows by a factor of 1.2 in every step. Both are equivalent but they are expressed from different points of view. We will use the terms difference equation, iteration process or recurrence relation interchangeably.

Some authors like to up the index by 1 and write

$$p(n+1) = p(n) + f(p(n))$$

while others prefer subscripts for indexing, and the equations become

$$p_n = p_{n-1} + f(p_{n-1}) = g(p_{n-1})$$

or

$$p_{n+1} = p_n + f(p_n) = g(p_n)$$

All of these describe the same process; they are packaged differently, but the mathematics remains the same.

### 16.2.1 Computers and Calculators

Explicit solutions to difference equation are the exception. There is a class of special cases (the affine equations) which admit solution formulas, but in most cases we have to settle for numerical results. We have seen before that these computations can be very tedious and time consuming and we turn to computing devices for a remedy.

The MODE setting for graphing calculators<sup>34</sup> often contains a SEQ option (SEQ stands for sequence). It is designed to construct a sequence of numbers from the iterative formula

$$u(n) = g(u(n-1))$$

Once you are in the SEQ mode, and you select "Y=" you will be asked for a number of things:

nMin	starting value of index.
u(n)	implement the formula here; use 2ND u (above the 7) for $u$ and the X,T,Θ,n key for $n$ .
u(nMin)	initial condition.

A plot of the solution can be obtained with the GRAPH key, and TABLE returns the solution in tabular form.

**Example:** We solve

$$\begin{aligned} a(n) &= a(n-1) \\ &\quad + 0.25a(n-1)(4 - a(n-1)) \\ a(0) &= 0.1 \end{aligned}$$

---

<sup>34</sup>This is in reference to TI calculators. Check user manuals for other brands.

numerically. Here the iteration function is

$$g(a) = a + 0.25a(1 - a)$$

Once SEQ has been selected, the entries in "Y=" should be

$$\begin{aligned} n_{\text{Min}} &= 0 \\ u(n) &= u(n-1) + 0.25 \cdot u(n-1) \cdot (4 - u(n-1)) \\ u(n_{\text{Min}}) &= \{ 0.1 \} \end{aligned}$$

The name of the variable is immaterial, and we have to live with the fact that we have to use  $u$  in place of  $a$ .

The TABLE command returns

$n$	$u(n)$
0	.1
1	.1975
2	.38525
3	.73339
4	1.3323
$\vdots$	$\vdots$

and with GRAPH we can visualize the result.

In EXCEL we can use relative cell references and the quick fill handle to obtain the same table. We stick to the same example, and the result is shown below.

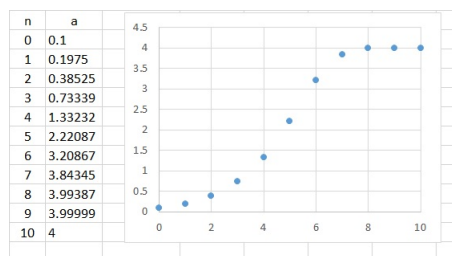
B9			B9		
A	B	C	A	B	C
1	n	a	1	n	a
2	0	0.1	2	0	0.1
3	1	0.1975	3	1	=B2+0.25*B2*(4-B2)
4	2	0.38524844	4	2	=B3+0.25*B3*(4-B3)
5	3	0.73339279	5	3	=B4+0.25*B4*(4-B4)
6	4	1.33231933	6	4	=B5+0.25*B5*(4-B5)
7	5	2.22086996	7	5	=B6+0.25*B6*(4-B6)
8	6	3.20867407	8	6	=B7+0.25*B7*(4-B7)
9	7	3.84345082	9	7	=B8+0.25*B8*(4-B8)
10	8	3.99387309	10	8	=B9+0.25*B9*(4-B9)
11	9	3.99999062	11	9	=B10+0.25*B10*(4-B10)
12	10	4	12	10	=B11+0.25*B11*(4-B11)
13			13		

The left part contains the EXCEL output, the right part displays the actual cell entries. The column with the index  $n$  is included for convenience, and the values of  $a(n)$  are shown in column B.

The cell B2 contains the initial value  $a(0) = 0.1$ . The cell B3 directly below it implements the recurrence formula and calculates  $a(1)$ . It starts with an "=", like all formulas in EXCEL do, and then it makes repeated reference to cell B2. In fact, we are entering  $g(B2)$ , that is, any occurrence of  $a$  in the formula for  $g$  is replaced by B2.

The next cell in the column is B4 and it contains  $a(2)$ . Its computation requires calculation of  $g(B3)$ , and we redo the last step with B2 replaced by B3. As we go down the column, we have to repeat this process over and over. Fortunately, we only have to enter the formula only once in B3; for the rest we use the fill feature<sup>35</sup>. The cell references will be updated automatically.

EXCEL makes it very easy to display the solutions graphically as a scatter plot.



## 16.3 Steady States and Stability

A steady state of a difference equation is a value of the state variable which does not change anymore.

**Example:** Consider (we have seen this example before)

$$a(n) = 0.6a(n-1) + 24$$

If  $a(0) = 60$ , then

$$a(1) = 0.6 \cdot 60 + 24 = 36 + 24 = 60$$

<sup>35</sup>Go to the lower right of the cell until a "+" appears. Then hold and drag.

and we see that  $a(n) = 60$  for all subsequent terms<sup>36</sup>.

A steady state is also called a fixed point or an equilibrium. It is a value  $p^*$  of the state variable such that

$$p(n) = p(n-1) = p^*$$

To find a steady state, replace both,  $p(n)$  and  $p(n-1)$ , by  $p$  and solve the resulting equation. The solutions  $p^*$  are the desired fixed points.

**Example:** Consider (we have used this example in the computer section)

$$\begin{aligned} p(n) &= p(n-1) \\ &\quad + 0.25p(n-1)(4 - p(n-1)) \end{aligned}$$

Setting  $p(n) = p(n-1) = p$  leads to

$$\begin{aligned} p &= p + 0.25 p (4 - p) \\ 0 &= 0.25 p (4 - p) \end{aligned}$$

and we have two equilibria, namely  $p^* = 0$  and  $p^* = 4$ .

Check: If  $p(0) = 0$ , then

$$p(1) = 0 + 0.25 \cdot 0 \cdot (4 - 0) = 0$$

and if  $p(0) = 4$ , then

$$p(1) = 0 + 0.25 \cdot 4 \cdot (4 - 4) = 4$$

If the difference equation has the form (25) we can argue that the update  $f(p)$  must vanish for steady states, and we can find fixed points by setting  $f(p) = 0$ .

**Example:**

$$\begin{aligned} p(n) &= p(n-1) \\ &\quad + 0.25p(n-1)(4 - p(n-1)) \end{aligned}$$

<sup>36</sup>Difference equations follow a domino effect. If there is no change in the step from  $n = 0$  to  $n = 1$ , then there are no changes in the later steps and  $a(n)$  will remain constant.

Here  $f(p) = 0.25 p (4 - p)$ , and  $p^* = 0$  and  $p^* = 4$  result in  $f(p^*) = 0$ .

Fixed points can be stable or unstable. Figure 37 shows solutions to

$$\begin{aligned} p(n) &= p(n-1) \\ &\quad + 0.25p(n-1)(4 - p(n-1)) \end{aligned}$$

with different starting values. When started exactly at  $p(0) = 0$  or  $p(0) = 4$ , the trajectories remain constant. The solutions starting at  $p(0) = \pm 0.1$  move away from zero, which makes  $p^* = 0$  an *unstable* equilibrium. On the other hand, solutions appear to be pulled toward  $p^* = 4$ , and we call this a *stable* steady state.

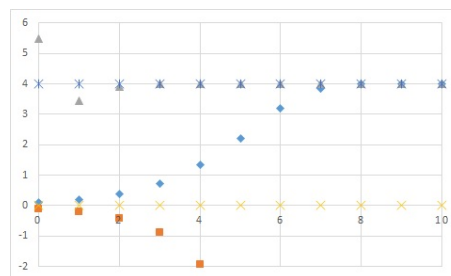


Figure 37: Stability Example

Can we predict stability without calculating iterates and turning to graphs?

To answer this question, we go back to the standard form of a difference equation

$$p(n) = p(n-1) + f(p(n-1))$$

If  $f(p(n-1)) > 0$ , then

$$p(n) > p(n-1)$$

and the values of  $p$  will increase. Conversely, if  $f(p(n-1)) < 0$ , then

$$p(n) < p(n-1)$$

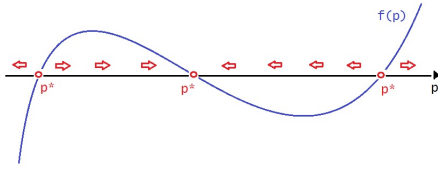


Figure 38: An Update Function

and  $p$  decreases. For  $f(p(n-1)) = 0$ , we find that  $p(n) = p(n-1) = p^*$ , and we are at a fixed point.

In the graph in Figure 38 depicts an example of an update function. The states  $p$  make up the horizontal axis (in our previous graphs,  $p$  was shown along the vertical axis). When the curve lies above the  $p$ -axis, the iterates  $p(n)$  become bigger, and the flow goes to the right; when the curve lies below the  $p$ -axis, the situation is reversed. The  $p$ -intercepts make up the steady states.

Still in reference to Figure 38, the steady states on the outside must be unstable: If  $p > p^*$ , the update is positive and  $p$  becomes even bigger and moves away from  $p^*$ . Conversely, if  $p < p^*$ , the update will be negative, and again  $p$  moves away from  $p^*$ . Therefore, a steady state will be unstable if  $f(p)$  changes from negative to positive at  $p^*$ , or in other words, a steady state  $p^*$  is unstable, if  $f$  has a positive slope<sup>37</sup> at  $p^*$ .

It is tempting to say that  $p^*$  is stable, if  $f(p)$  has a negative slope at  $p^*$ . But it is not that simple. If  $p < p^*$  and  $f(p) > 0$ , then  $p$  will increase. But if  $f$  is very steep, and  $f(p)$  is very large, the increase may be so large that we overshoot  $p^*$  by a lot and move further away

<sup>37</sup>We use this term loosely. Straight lines have the same slope everywhere, for other curves the slopes vary. Calculus defines the slope of a function  $f(x)$  at a particular point  $x$  as the derivative, denoted by  $f'(x)$ , and a major portion of Calculus is devoted to the study of derivatives.

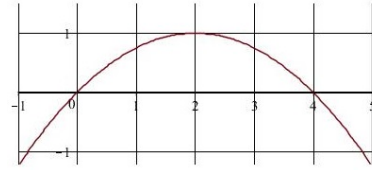
from it, however, on the other side from where we started. As it turns out,  $p^*$  is stable, if  $f(p)$  has a negative slope at  $p^*$ , and the slope is between  $-2$  and  $0$ .

A careful sketch of  $f$  will reveal the slope at any point, and as a rule of thumb, be cautious if the update function is steep at a fixed point.

**Example:** We return to

$$p(n) = p(n-1) + 0.25p(n-1)(4 - p(n-1))$$

where  $f(p) = 0.25 p (4 - p)$ . The graph of  $f$  becomes



The two fixed points are located at the the points where the curve intersects with the horizontal axis ( $p^* = 0$  and  $p^* = 4$ ).  $f(p)$  is positive between  $p = 0$  and  $p = 4$ . This is the region where the points  $p(n)$  are increasing (look at the labels on the vertical axis in Figure 37). Outside that range  $f$  is negative and the iterates decrease.

At the point  $p^* = 0$ , the graph of  $f$  switches from negative to positive, and the equilibrium is unstable. At  $p^* = 4$ ,  $f$  changes from positive to negative, and the slope is  $-1$ , as a careful sketch reveals. Thus,  $p^* = 4$  is stable.

**Example:**

$$p(n) = 1.1p(n-1) - 100$$

This is the opening example (23) of the deer population.

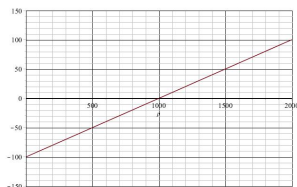
First we have to identify  $f$ . Subtracting  $p(n-1)$  on both sides results in

$$p(n) - p(n-1) = 0.1p(n-1) - 100$$

Hence, (replace  $p(n-1)$  by  $p$ )

$$f(p) = 0.1p - 100$$

The equation  $f(p) = 0$  has solution  $p^* = 1,000$ . When 1,000 deer are present, a 10% increase leads to 100 more animals, which is exactly the amount lost to hunting, and the population will remain constant.



$f$  is a line with a positive slope, and therefore the steady state is unstable.

In the original example we had the initial count  $p(0) = 1,200$ , and the iterates move away from  $p^* = 1,000$  toward positive infinity.

If we use  $p(0) = 800$  as initial condition, we find that

$$\begin{aligned} p(1) &= 1.1 \cdot 800 - 100 = 780 \\ p(2) &= 1.1 \cdot 780 - 100 = 758 \\ p(3) &= 1.1 \cdot 758 - 100 = 733.8 \end{aligned}$$

and we see that the iterates move away from  $p^* = 1,000$ , this time toward negative infinity, and the model becomes meaningless once the population falls below zero (extinction).

**Example:** Here we explore what happens when  $f$  has a negative slope, but it is too steep. Let

$$f(p) = -2.5p + 10$$

$f(p) = 0$  leads to  $p^* = 4$ , and because the slope of  $f$  is less than  $-2$ , this steady state is unstable.

Suppose that we start very close to  $p^*$  with  $p(0) = 3.9$ . Then

$$\begin{aligned} f(3.9) &= -2.5 \cdot 3.9 + 10 = 0.25 \\ p(1) &= 3.9 + f(3.9) = 4.15 \\ f(4.15) &= -2.5 \cdot 4.15 + 10 = -0.375 \\ p(2) &= 4.15 - 0.375 = 3.775 \end{aligned}$$

and we see that the iterates move away from  $p^* = 4$  in an alternating fashion.

A lot more can be said about stability with the tools of Calculus. But this is beyond the scope of this workbook.

### 16.3.1 Affine Equations

This section is all math. Its main purpose is to give a little background for an important class of difference equations. It can be used as reference material or as additional reading for the interested student. The remainder of the workbook will not make use of the material in this section.

It is possible to find exact solution formulas for the class of affine difference equations. We shall present these formulas, and show how they can be derived.

**Definition:** Difference equations of the form

$$p(n) = p(n-1) + r p(n-1) + k \quad (26)$$

are called *affine*. Here  $r$  and  $k$  are constants.

In this case the associated update function

$$f(p) = r p + k$$

is linear, with slope  $r$  and intercept  $k$ .

#### Special Case: Linear Growth

Here  $r = 0$ , and the difference equation becomes

$$p(n) = p(n-1) + k \quad (27)$$

that is, we add  $k$  at each iteration.

The solution is given by

$$p(n) = kn + p(0)$$

which is a linear function for the variable  $n$  with slope  $k$  and intercept  $p(0)$ .

The verification of this formula requires two steps. First we check the initial condition, and it is plain that for  $n = 0$  we obtain  $p(0) = p(0)$ . Next, we test the difference equation. We begin on the right of (27), substitute the solution with  $n$  replaced by  $n - 1$ , and simplify:

$$\begin{aligned} & p(n-1) + k \\ &= k(n-1) + p(0) + k \\ &= kn - k + p(0) + k = kn + p(0) \\ &= p(n) \end{aligned}$$

and we are done.

Steady states are redundant. Since  $f(p) = k$ , equilibria are only possible if  $k = 0$ . But then

$$p(n) = p(n-1)$$

and any value is an equilibrium, because nothing changes.

### Special Case: Exponential Growth.

Here we consider the case where  $k = 0$ , which results in

$$p(n) = p(n-1) + r p(n-1) \quad (28)$$

The update function becomes  $f(p) = r p$ , which is a line of slope  $r$  passing through the origin.

The solution is given by

$$p(n) = p(0)(1+r)^n \quad (29)$$

which is an exponential function with multiplier  $M = 1 + r$ .

For  $n = 0$  we have  $p(0)$  on both sides, and the initial condition is confirmed. Again, beginning on the right of (28), we substitute the

solution and simplify:

$$\begin{aligned} & p(n-1) + r p(n-1) \\ &= (1+r) p(n-1) \\ &= (1+r) p(0) (1+r)^{n-1} \\ &= p(0) (1+r)^n \\ &= p(n) \end{aligned}$$

This shows that the function (29) satisfies the difference equation (28).

The update function is

$$f(p) = r p$$

The case  $r = 0$  is redundant. For  $r \neq 0$  it follows that  $p^* = 0$  is the only steady state. It is stable if and only if the slope satisfies  $-2 < r < 0$ .

The **general affine equation** (26) has the update function  $f(p) = r p + k$ , and setting  $f(p) = 0$  leads to

$$p^* = -\frac{k}{r}$$

as the only fixed point. It is stable if  $-2 < r < 0$ , because then the slope of  $f$  is in the desired range.

Our next goal is to derive a solution formula, and we use a common technique (trick) of applied math: Instead of using the population values  $p$  directly, we use a new variable  $y$  to measure by how much we deviate from the equilibrium  $p^*$ . The relationship becomes

$$p = y + p^* \quad \text{or} \quad y = p - p^*$$

and we use whichever formula is more suiting.

$p$  follows a difference equation, and we construct the corresponding equation for  $y$ :

$$\begin{aligned} & y(n) \\ &= p(n) - p^* \\ &= p(n-1) + r p(n-1) + k - p^* \end{aligned}$$

$$\begin{aligned}
&= y(n-1) + p^* \\
&\quad + r(y(n-1) + p^*) + k - p^* \\
&= y(n-1) + r y(n-1) + r \left(-\frac{k}{r}\right) + k \\
&= y(n-1) + r y(n-1)
\end{aligned}$$

Thus,  $y(n)$  follows an exponential growth model, and we already found a solution formula for this case. Using (29) we have

$$y(n) = y(0)(1+r)^n$$

and back-substitution yields

$$\begin{aligned}
&p(n) \\
&= y(n) + p^* \\
&= y(0)(1+r)^n + p^* \\
&= (p(0) - p^*)(1+r)^n + p^* \\
&= p(0)(1+r)^n + p^*(1 - (1+r)^n)
\end{aligned}$$

and thus

$$p(n) = p(0)(1+r)^n + p^*(1 - (1+r)^n) \quad (30)$$

When  $-2 < r < 0$ , we have  $-1 < (1+r) < 1$ , and high powers of this number will become very small. In this case, the terms  $(1+r)^n$  become negligible, and formula (30) shows that  $p(n) \approx p^*$ , no matter where we start initially. Again, this confirms that the steady state is stable.

We review older examples using the new formula (30).

**Example:** In the introductory section we looked at deer populations following the difference equation

$$p(n) = p(n-1) + 0.1p(n-1) - 100$$

This is an affine equation with  $r = 0.1$  and  $k = -100$ . The steady state is

$$p^* = -\frac{k}{r} = 1,000$$

It is unstable, because  $r > 0$ .

We had the initial condition  $p(0) = 1,200$  and formula (30) yields

$$p(n) = 1,200 \cdot 1.1^n + 1,000 \cdot (1 - 1.1^n)$$

For example, if we are interested in  $p(3)$  we find that

$$\begin{aligned}
p(3) &= 1,200 \cdot 1.1^3 + 1,000 \cdot (1 - 1.1^3) \\
&= 1597.2 - 331 = 1,266.2
\end{aligned}$$

without having to compute  $p(1)$ ,  $p(2)$  and  $p(3)$  successively.

If the initial condition is changed to  $p(0) = 800$ , as we did later in the stability discussion, and we still are interested in  $p(3)$ , we get

$$\begin{aligned}
p(3) &= 800 \cdot 1.1^3 + 1,000 \cdot (1 - 1.1^3) \\
&= 1597.2 - 331 = 733.8
\end{aligned}$$

**Example:** The new car loan study falls into the category of affine equations. Here we had

$$B(n) = B(n-1) + 0.004B(n-1) - 400$$

with  $B(0) = 10,000$ . In this case

$$f(B) = 0.004B - 400$$

and  $r = 0.004$  and  $k = -400$ .

The steady state is  $B^* = -\frac{-400}{0.004} = 100,000$  and we have

$$B(n) = 10,000 \cdot 1.004^n + 100,000 \cdot (1 - 1.004^n)$$

So, for instance, the remaining debt after one year is

$$\begin{aligned}
&B(12) \\
&= 10,000 \cdot 1.004^{12} + 100,000 \cdot (1 - 1.004^{12}) \\
&= 10,490.70 - 4907.02 = 5,583.68
\end{aligned}$$

What is the significance of the steady state? For one, it is unstable, because  $r > 0$ . Secondly, if the balance for some reason equals



\$100,000 exactly, then a \$400 payment covers the interest only, and the debt remains the same. The person who owes more than \$100,000, will get deeper in debt with just \$400 payments; the person who owes less than \$100,000 will eventually be debt-free with regular \$400 installments.

**Example:**

$$a(n) = 0.6 a(n-1) + 24 \quad a(0) = 0$$

This example was used in graphing and in the stability discussion.

When we subtract  $a(n-1)$  on both sides, we find that

$$a(n) - a(n-1) = -0.4 a(n-1) + 24$$

Therefore,

$$f(a) = -0.4a + 24$$

and  $r = -0.4$  and  $k = 24$ . The equilibrium

$$a^* = -\frac{24}{-0.4} = 60$$

is stable, because  $-2 < r < 0$ . Formula (30) implies that

$$a(n) = 60 (1 - 0.6^n)$$

In the graphing example the value  $a(10)$  was found in a successive computation. Our formula yields

$$\begin{aligned} a(10) &= 60 (1 - 0.6^{10}) \\ &= 60 (1 - 0.006,065) = 59.64 \end{aligned}$$

which, of course, coincides with the value we have found before.

## 16.4 Logistic Growth

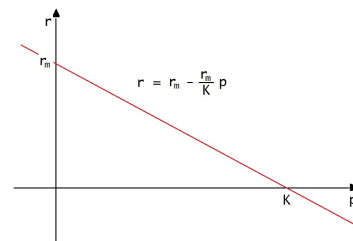
The exponential model

$$p(n) = p(n-1) + r p(n-1)$$

is a good way to describe population growth, but it suffers from a big handicap: The populations will always increase, at the same constant rate, and they will grow without bound. This is not realistic. As a population gets larger, the fight for resources will become more intense, and the growth should slow down.

The logistic growth model addresses this situation. It uses a population dependent growth rate with two important parameters, the maximum growth rate  $r_m$  and the carrying capacity  $K$ .

The maximum growth rate  $r_m$  is the rate associated with unlimited resources. This rate applies when the population is rather small. The other parameter is the carrying capacity  $K$ . This is the maximum population size which the environment can support. If the population gets past this point, it will encounter a negative growth rate and it will decline.



A linear function is the simplest way to accommodate this description. We use

$$r = r_m - \frac{r_m}{K} p = r_m \left(1 - \frac{p}{K}\right)$$

When  $p = 0$ , we get  $r = r_m$ , and at carrying capacity we have  $p = K$  and  $r = 0$ .

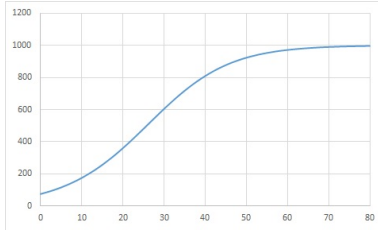
Upon substitution, the former exponential model now becomes the logistic model

$$p(n) = p(n-1) + r_m \left(1 - \frac{p(n-1)}{K}\right) p(n-1) \quad (31)$$

**Example:** With  $r_m = 0.1$  and  $K = 1,000$ , the logistic model (31) has the form

$$p(n) = p(n-1) + 0.1 \left(1 - \frac{p(n-1)}{1,000}\right) p(n-1)$$

The graph shows the solution with  $p(0) = 75$  as initial condition



The points are so close that the scatter plot looks like a smooth curve. We also observe the typical S-shape of logistic growth curves.

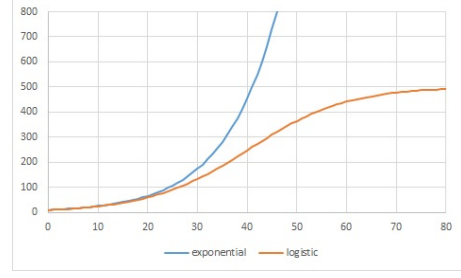
**Example:** Here we contrast the exponential versus the logistic model. We use  $r_m = 0.1$ ,  $K = 500$  and  $p(0) = 10$ . The exponential model is

$$p(n) = p(n-1) + 0.1 p(n-1)$$

while the logistic model becomes

$$\begin{aligned} p(n) &= p(n-1) + 0.1 \left(1 - \frac{p(n-1)}{500}\right) p(n-1) \\ &= p(n-1) + 0.01 p(n-1) - \frac{p(n-1)^2}{5,000} \end{aligned}$$

Both solutions are displayed in the figure below.



We can clearly see that the exponential model grows toward infinity, while the logistic model approaches the carrying capacity  $K = 500$ . In the early stages, until about  $n = 20$ , both curves are pretty much identical, at least it appears as such in the graph, and we call it the *exponential growth phase* of logistic growth. Here the population size  $p$  is fairly small, compared to the carrying capacity, and the population grows as if the resources were unlimited.

The fastest growth in the logistic model occurs when the population reaches  $p = 250 = \frac{1}{2} K$ . This is called the inflection point. Thereafter the growth decelerates and it will approach zero as the population approaches the steady state  $K = 500$ .

It follows from formula (31), that the update function for logistic growth is

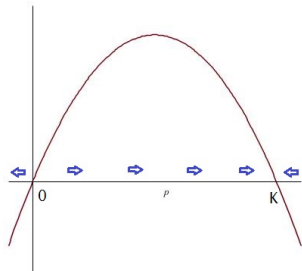
$$\begin{aligned} f(p) &= r_m \left(1 - \frac{p}{K}\right) p \\ &= r_m p - \frac{r_m p^2}{K} \end{aligned}$$

just replace  $p(n-1)$  by the variable  $p$ . In this regard, logistic growth is exponential growth ( $f(p) = r_m p$ ) with a quadratic correction term ( $-\frac{r_m p^2}{K}$ ). If  $p$  is fairly small compared to the carrying capacity, we do not have to worry about this term, but as the populations increases, the quadratic term becomes more and more relevant.

**Stability.** We see from the factored form of the update function that

$$f(0) = 0 \quad \text{and} \quad f(K) = 0$$

which shows that logistic growth has steady states  $p^* = 0$  and  $p^* = K$ . The graph of  $f$  is a parabola opening downward. It increases at  $p^* = 0$ , which makes this steady state unstable.  $f$  decreases at  $p^* = K$ , and the this steady state is stable, unless  $f$  is too steep<sup>38</sup>.



The fastest growth occurs when  $p$  is one half of the carrying capacity.

At carrying capacity the population has reached the maximum attainable population size. This is a state of fierce competition and fight for survival, as the environment cannot support additional organisms.

**Epidemics.** Logistic growth is not limited to population models. In our next example we look at a model for the spread of a contagious disease. By  $p$  we denote the proportion of infected individuals. Then the uninfected proportion is  $q = 1 - p$ , because the infected plus the uninfected proportions have to add to one. The disease is spreading whenever infected individuals come into contact with not infected individuals, and the increase of infected individuals is proportional to the product  $pq$ . This, by the way, is a from of the mass-action law. The update function then becomes

$$f(p) = r p q = r (1 - p) p$$

<sup>38</sup>The carrying capacity  $K$  is stable, if  $0 < r_{max} < 2$ . Periodic, yet stable, patterns appear for  $1 < r < 2$  and cyclic behavior and chaotic patterns evolve if  $2 \leq r_{max} < 3$ .

for some constant  $r$ . This is logistic growth for  $K = 1 = 100\%$ , and the iterations become

$$\begin{aligned} p(n) \\ = p(n-1) + r (1 - p(n-1)) p(n-1) \end{aligned}$$

In this model all individuals will eventually become infected, because  $p$  will always approach carrying capacity in logistic growth. The model assumes that interactions between individuals are random, it does not allow for recovery from the disease, and it does not distinguish between susceptible individuals and those who have built up an immunity. But, despite of all of its shortfalls, it is a good starting point for the study of epidemics.

#### 16.4.1 Logistic Approximations

Collected data may show a logistic trend, but since we do not have an explicit formula for the logistic iterates, it makes it difficult to determine a suited approximation.

Calculus uses the curve

$$p = \frac{K}{1 + ae^{-rx}}$$

to describe logistic growth. Here  $a$  has to be selected to match the initial condition at  $x = 0$

$$p(0) = \frac{K}{1 + a}$$

and  $e$  is the Euler constant  $e = 2.718, 281, \dots$

The formula for  $p$  is found by solving the logistic differential equation<sup>39</sup>. While not an exact solution of the logistic difference equation (31), it is a fairly good approximation.

Unfortunately, logistic approximation is not a trendline option in EXCEL, but the graphing calculator has this capability: Enter the

<sup>39</sup>This is the equation (Calculus required)

$$\frac{dp}{dx} = r \left(1 - \frac{p}{K}\right) p$$

data into lists with STAT and EDIT, as usual. Then use STAT and CALC, and select "Logistic". The calculator will provide values for  $K$ ,  $a$  and  $r$  in the logistic function.

**Example:** Consider the data

$n$	$p$
0	2
1	3
2	5
4	12
6	18
10	24

We enter the values into two lists on a calculator, and select the logistic approximation as described above. The calculator responds with

$$y = c/(1 + ae^{(-bx)})$$

$$a = 12.17918098$$

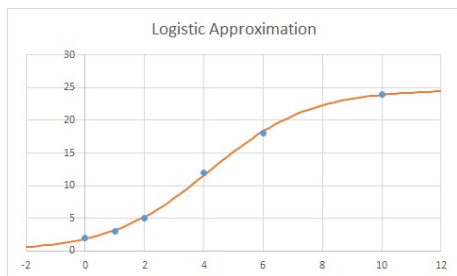
$$b = .5934057215$$

$$c = 24.66931882$$

and the logistic approximation of the data becomes

$$p = \frac{24.669}{1 + 12.179e^{-0.59341x}}$$

A graph of the points with the approximating curve is given below.



Rounded to three significant digits, the calculator found  $K = c = 24.669$  as carrying capacity; for the rate we have  $r_m = b = 0.593$ ,

and with  $x = 0$  we see that  $p(0)$  becomes

$$p(0) = \frac{24.669}{1 + 12.179} = 1.87$$

The solution to the associated difference equation

$$p(n) = p(n-1) + 0.593 \left( 1 - \frac{p(n-1)}{24.7} \right) p(n-1)$$

with  $p(0) = 1.87$ , along with the original data is given below. The approximations are a good match at the ends, but there are some gaps in the center.

$n$	$p(n)$	data
0	1.87	2
1	2.89	3
2	4.41	5
3	6.56	
4	9.42	12
5	12.87	
6	16.53	18
7	19.77	
8	22.11	
9	23.48	
10	24.17	24

## 16.5 Worked Problems

- Find a difference equation for the given situation, and calculate the first three iterates. Find steady states, if possible, and address their stability.
  - You work through a math book at 6 pages per day, beginning on page 23.
  - Beginning with \$200, a child spends half of its money in any given week, while also getting a \$10 allowance.

- (c) A fish population in a lake decreases at an annual rate of 5%. To combat this decline, 400 fish are released each year. In the beginning there are 5,000 fish in the lake.
- (d) Bacteria grow at a maximum rate of 3% per minute. We start with 25 bacteria, and the carrying capacity is 10,000. Use a logistic model.

Solutions:

- (a) If  $p$  denotes the starting page, we know that  $p(0) = 23$ , and the update function is  $f(p) = 6$ . The difference equation then takes the form

$$p(n) = p(n-1) + 6$$

and the first iterates become

$$p(1) = p(0) + 6 = 23 + 6 = 29$$

$$p(2) = p(1) + 6 = 29 + 6 = 35$$

$$p(3) = p(2) + 6 = 35 + 6 = 41$$

There are no steady states, because  $f(p) = 6 \neq 0$ .

- (b) We let  $p(n)$  be the available funds at the beginning of week  $n$ . Then  $p(0) = 200$ . For the update we find

$$f(p) = -\frac{1}{2}p + 10$$

This results in

$$\begin{aligned} p(n) &= p(n-1) - \frac{p(n-1)}{2} + 10 \\ &= \frac{p(n-1)}{2} + 10 \end{aligned}$$

The story of the first three weeks is

$$p(1) = \frac{200}{2} + 10 = 110$$

$$p(2) = \frac{110}{2} + 10 = 65$$

$$p(3) = \frac{65}{2} + 10 = 42.50$$

The money disappears quickly, but the steady influx of \$10 keeps the child from being broke. The steady state is found from  $f(p) = 0$ , and we see that

$$p^* = 20$$

At this level the child spends \$10 each week, and get the money back in form of the allowance.  $f(p)$  is a line with a negative slope, which is between  $-2$  and  $0$ , and the steady state is stable.

- (c) Let  $p(n)$  be the fish population in year  $n$ . Then  $p(0) = 5,000$ . The update function is

$$f(p) = -0.05p + 400$$

The equation  $f(p) = 0$  yields

$$p^* = \frac{400}{0.05} = 8,000$$

When 8,000 fish are present, a 5% decline means the loss of 400 fish, which is balanced by restocking. The equilibrium is stable, because  $f$  is a line with a negative slope between  $-2$  and  $0$ .

The difference equation becomes

$$\begin{aligned} p(n) &= p(n-1) - 0.05p(n-1) + 400 \\ &= 0.95p(n-1) + 400 \end{aligned}$$

and for the first iterations we have

$$p(1) = 0.95 \cdot 5,000 + 400 = 5,150$$

$$p(2) = 0.95 \cdot 5,150 + 400 = 5,292.5$$

$$p(3) = 0.95 \cdot 5,292.5 + 400 = 5,427.9$$

- (d) This is a logistic growth problem.  
The information translates into

$$\begin{aligned}r_m &= 0.03 \\K &= 10,000 \\p(0) &= 25\end{aligned}$$

and the equation becomes

$$\begin{aligned}p(n) &= p(n-1) + \\&\quad 0.03 \left(1 - \frac{p(n-1)}{10,000}\right) p(n-1)\end{aligned}$$

The next iterates are (calculator)

$$\begin{aligned}p(1) &= 25.748 \\p(2) &= 26.519 \\p(3) &= 27.312\end{aligned}$$

This is still in the exponential growth phase (25 is much less than 10,000), and the bacteria grow at 3% per minute. Incidentally,  $p(24) = 50.7$  and initially the doubling time is a little under 24 minutes.

The growth function is

$$f(p) = 0.03 \left(1 - \frac{p}{10,000}\right) p$$

with steady states  $p^* = 0$  (unstable) and  $p^* = 10,000$  (stable).

2. Compute  $a(5)$  from

$$\begin{aligned}\text{(a)} \quad a(n) &= 0.8 \cdot a(n-1) \\a(0) &= 250\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad a(n) &= \frac{a(n-1)}{2} + 20 \\a(0) &= 80\end{aligned}$$

and find a general formula for  $a(n)$ .

Solutions

- (a) Here we multiply by 0.8 repeatedly.

$$\begin{aligned}a(1) &= 0.8 \cdot 250 = 200 \\a(2) &= 0.8 \cdot 200 = 160 \\a(3) &= 0.8 \cdot 160 = 128 \\a(4) &= 0.8 \cdot 128 = 102.4 \\a(5) &= 0.8 \cdot 102.4 = 81.92\end{aligned}$$

The formula for the iterates is

$$a(n) = 250 \cdot 0.8^n$$

This follows from formula (29) with  $r = -0.2$ .

- (b) Here the rule is: Beginning with 80, take half of the current value and add 20. This results in

$$\begin{aligned}a(1) &= \frac{80}{2} + 20 = 60 \\a(2) &= \frac{60}{2} + 20 = 50 \\a(3) &= \frac{50}{2} + 20 = 45 \\a(4) &= \frac{45}{2} + 20 = 42.5 \\a(5) &= \frac{42.5}{2} + 20 = 41.25\end{aligned}$$

By looking at the data, we notice that in each step we cut the distance to  $a^* = 40$  in half, and the general formula becomes

$$a(n) = 40 + \frac{40}{2^n}$$

We could also use formula (30) with  $a^* = 40$ ,  $r = -0.5$  and  $a(0) = 80$ :

$$\begin{aligned}a(n) &= 80 \cdot \left(\frac{1}{2}\right)^n + 40 \cdot \left(1 - \left(\frac{1}{2}\right)^n\right) \\&= 40 + (80 - 40) \cdot \left(\frac{1}{2}\right)^n \\&= 40 + \frac{40}{2^n}\end{aligned}$$

3. Find a difference equation for the data in the table.

	$n$	$p(n)$
	0	65
(a)	1	69
	2	73
	3	77
	4	81

	$n$	$p(n)$
	0	80
(b)	1	120
	2	180
	3	270
	4	405

	$n$	$p(n)$
	0	81
(c)	1	54
	2	36
	3	24
	4	16

Solutions: In this type of problem you have to identify a pattern in the table, and then translate it into an update function and a difference equation. Nothing can be done if you don't see the pattern.

- (a) Here the values go up by 4 in each step. The update is  $f(p) = 4$  and the difference equation becomes

$$p(n) = p(n-1) + 4$$

- (b) We add one half of the number each time. This makes  $f(p) = p/2$  and we have

$$\begin{aligned} p(n) &= p(n-1) + \frac{p(n-1)}{2} \\ &= 1.5 p(n-1) \end{aligned}$$

- (c) Here we take 2/3 of the current number, and the equation becomes

$$p(n) = \frac{2}{3} p(n-1)$$

The update is  $f(p) = -p/3$ , because if 2/3 are left, then 1/3 must have been taken out.

4. Consider the update function

$$f(p) = \frac{p(p-4)(10-p)}{50}$$

Compute  $p(3)$  for the initial values

(a)  $p(0) = 3$

(b)  $p(0) = 4$

(c)  $p(0) = 5$

Solutions: In all cases we compute the update  $f(p)$  first, and then determine the next  $p$ .

(a)  $f(3) = \frac{3(-1) \cdot 7}{50} = -\frac{21}{50} = -0.525$

Therefore

$$p(1) = 3 - 0.525 = 2.475$$

Following the same pattern we obtain

$$f(2.475) = -0.710$$

$$p(2) = 2.475 - 0.710 = 1.765$$

$$f(1.765) = -0.812$$

$$p(3) = 1.765 - 0.812 = 0.953$$

- (b) We have  $f(4) = 0$ , which makes  $p^* = 4$  an equilibrium point, and

$$a(0) = a(1) = a(2) = a(3) = 4$$

- (c) Routine calculations result in

$n$	$p(n)$	$f(p(n))$
0	5	0.625
1	5.625	1.000
2	6.625	1.467
3	8.092	

5. Find the equilibria when

(a)  $p(n) = 0.25p(n-1) + 300$

(b)  $p(n) = 1.2p(n-1) - 80$

(c)  $p(n) = p(n-1) + 0.4 \left(1 - \frac{p(n-1)}{250}\right) p(n-1)$

and discuss their stability.

Solutions:

- (a) If we substitute  $p = p(n) = p(n-1)$ , we find that

$$p = 0.25p + 300$$

which results in the solution  $p^* = 400$  as steady state. Is it stable? First, we determine update function. Subtraction yields

$$\begin{aligned} p(n) - p(n-1) \\ = -0.75p(n-1) + 300 \end{aligned}$$

Hence,

$$f(p) = -0.75p + 300$$

The update function is linear with slope between  $-2$  and  $0$ , and therefore the steady state is stable.

- (b) We begin by finding the update:

$$\begin{aligned} p(n) - p(n-1) \\ = 0.2p(n-1) - 80 \end{aligned}$$

Thus,  $f(p) = 0.2p - 80$ . The steady state is found from  $f(p) = 0$ , with solution  $p^* = 400$ .  $f$  has a positive slope, therefore the steady state is unstable.

- (c) This is a logistic growth problem. The steady states are  $p^* = 0$  and  $p^* = 250$ .

In logistic growth,  $p^* = 0$  always is an unstable steady state, because if only very few organisms are present, the population will start growing and thus move away from  $p^* = 0$ .

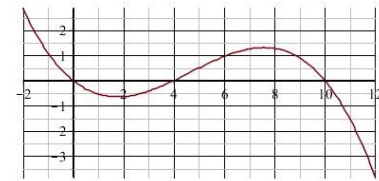
The maximum growth rate in this example is  $r_{max} = 0.4$ , and the carrying capacity is stable, because  $0 < r_{max} < 2$ .

6. Find the steady states for the update function

$$f(p) = \frac{p(p-4)(10-p)}{50}$$

and discuss stability.

Solution: Steady states are solutions of  $f(p) = 0$ , and thus we have  $p^* = 0$ ,  $p^* = 4$  and  $p^* = 10$  as equilibria. The graph of  $f$  is given below.



The slopes at  $p^* = 0$  and  $p^* = 10$  are negative, and above  $-2$ . Therefore these points are stable. At  $p^* = 4$  we have a positive slope, and this steady state is unstable.

This update function was used above. We saw that  $p^* = 4$  was a steady state, and when the iterations were started at  $p(0) = 3$ , the iterates moved away toward  $0$ , while for  $p(0) = 5$  the iterates moved toward  $10$ . This illuminates the unstable nature of the equilibrium.

7. **Epidemics.** An epidemic spreads according to the law

$$p(n)$$



$$= p(n-1) + 0.6(1-p(n-1))p(n-1)$$

where  $n$  measures time in months, and  $p(n)$  is the proportion of infected individuals in the  $n$ -th month. Initially 3% of the population is infected. How many will be infected after 4 month? How many after one year?

Solution: This problem is purely computational.  $p(0) = 0.03$  is known, and we are asked to compute  $p(4)$  and  $p(12)$ .

$n$	$p(n)$
0	0.030
1	0.047
2	0.075
3	0.116
4	0.178
5	0.265
6	0.382
7	0.524
8	0.673
9	0.805
10	0.899
11	0.954
12	0.980

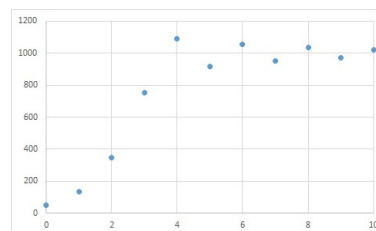
We see that after four months about 18% are infected, and that after one year 98% of the population carry the disease.

8. It was stated in the discussion of logistic growth that the carrying capacity is stable if  $0 < r_{max} < 2$ . In this problem we compute a few iterations when  $r_{max}$  is close to 2.

Compute  $p(1)$  though  $p(10)$  for logistic growth with  $r_{max} = 1.8$ ,  $K = 1000$  and  $p(0) = 50$ , and graph the result.

Again, this is a purely computational problem. The first values are  $p(1) = 135.5$ ,  $p(2) = 346.4$  and  $p(3) = 753.9$ . Because

$r_{max} = 1.8 > 1$  we will overshoot the carrying capacity, then bounce back below it, and keep oscillating about  $K = 1,000$  and gradually close in on this value.



## 16.6 Exercises

1. Find  $p(5)$  from

(a)  $p(n) = p(n-1) + 4$ ,  $p(0) = 25$

(b)  $p(n) = 1.1p(n-1)$ ,  $p(0) = 60$

(c)  $p(n) = 2p(n-1) - 10$ ,  $p(0) = 12$

2. Find  $p(5)$  from  $p(n) = \frac{1}{2}p(n-1) + 3$  and

(a)  $p(0) = 36$

(b)  $p(0) = 0$

(c)  $p(0) = 6$

3. Find a difference equation for the values in the table and fill in the missing values.

$n$	$p(n)$	$n$	$p(n)$
0	28	0	81
1	25	1	108
2	22	2	144
3	19	3	192
4		4	
5		5	

4. Find the equilibria for the difference equations, and discuss their stability.

(a)  $p(n) = 1.05p(n-1) - 200$

(b)  $p(n) = 0.75p(n-1) + 68$

$$(c) \quad p(n) = p(n-1) + 0.03 \left(1 - \frac{p(n-1)}{12}\right) p(n-1)$$

5. **Spread of a Disease.**  $p(n)$  is the percentage of infected people at time  $n$ , the quantity  $1 - p(n)$  is the percentage of not-infected people. The disease spreads whenever infected and uninfected people come in contact; this is represented by the product  $p(n-1)(1 - p(n-1))$ , and we arrive at the difference equation

$$p(n) = p(n-1) + p(n-1)(1 - p(n-1))$$

Given that initially 1% of the population are infected, what percentage is infected at time  $n = 5$ , what percentage when  $n = 10$ ?

6. Use a calculator to find a logistic approximation for the data

$x$	$p$
0	3
3	15
8	80
12	117
15	123

$n$	$p(n)$
0	28
1	25
2	22
3	19
4	16
5	13

$$p(n) = \frac{4}{3} p(n-1)$$

$n$	$p(n)$
0	81
1	108
2	144
3	192
4	256
5	341.3

4. (a)  $p^* = 4,000$ , stable  
 (b)  $p^* = 272$ , stable  
 (c)  $p^* = 0$ , unstable and  $p^* = 12$ , stable

5. 27.5% and 99.997%

$$6. \quad p = \frac{125}{1 + 35.4e^{-0.518x}}$$

$$r_{max} = 0.5183, K = 125.0, p(0) = 3.434$$

## Answers

- (a) 45  
 (b) 96.6  
 (c) 74
- (a) 6.9375  
 (b) 5.8125  
 (c) 6
- $p(n) = p(n-1) - 3$