Mathematical Modeling Via Multiple Representations

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Annual Meeting for the Society for Mathematical Biology, 2019
Outline

Rule-of-five Framework
- QUBES
- NIMBioS Black Box Working Group

Resource-Limited Growth: An Example of Mathematical Modeling via Representations
- An Experiential Representation
- A Numerical Representation
- Three Visual Representations
- A Verbal Representation
- A Symbolic Representation
- Computer Implementation
Resource Hub - qubeshub.org is designed to support teaching at the interface of mathematics and biology no matter where you are

Supported by NSF IUSE, NSF DBI, BIO SIGMAA, NCTM, SIAM, and COMAP and partner contracts
Run Peer Faculty Mentoring Networks, cloud based software, partner with professional societies and curriculum projects, organize conversations (like this one!).

This takes an understanding of best practices in teaching at the interface of mathematics and biology.

Proposed a working group to NIMBioS (National Institute for Mathematical Biology and Synthesis) to investigate and synthesize across domains how best to unpack the black box of models for biology students/teachers.
Interdisciplinary Team:
1 math ed, 3 bio ed, 1 bio SoTL, 2 math SoTL, 1 bio faculty dev
2 biologists, 2 math bio, 3 mathematicians, 1 physics bio ed (blurry lines)
Discussion of issues:

- What quantitative skills are needed?
- What tools are available for assessment?
- Math anxiety (instructor and student)
- Language

Additional research → language as key.

**Model/modeling** - as the mathematicians were thinking about constructing an equation from a concept and the biologists were thinking about fitting an equation to data.

Reflects other work done in physics for biology by Joe Redish
Models and Modeling

Definition
A model is a simplified, abstract or concrete representation of relationships and/or processes in the real world, constructed for some purpose.

Model Representations:
- Verbal
- Visual
- Symbolic
- Numerical

May know this as the "Rule-of-4" in calculus reform.

"The rate of population growth is proportional to its current size."

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Models and Modeling

**Definition**

**Modeling** is the act of moving between representations/models (arrow), checking model with reality and/or revising model.

**Modeling activities**

- Moving between representations/models (arrow)
- Checking model with reality
- Creating and revising model

**Modeling process**

- A set of modeling activities from reality to “good enough.”
- Reality & experiential are key!
- Defined to include approaches like data science
The Challenge

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- So how do we implement this theory in the classroom?
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- So how do we implement this theory in the classroom?
  - With classroom projects that ‘model’ modeling with a directed sequence of modeling activities.
Modeling Activity Objectives

1. Illustrate the true nature of science
   - Theory without observation (natural and/or experimental) is mere speculation.
   - Observation without theory is just a collection of data.
   - **Scientific progress is due to the combination of theory and observation.**
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2. Provide a rich experience of mathematical modeling
   ○ Use all five representations and make many connections.
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   - Models are not depictions of reality; they are abstractions that under best circumstances have explanatory value.
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4. Teach the principles of density-dependent growth
Approach

- The real world is complicated.
  - Hard to collect data.
  - Many confounding complications.
  - Demographic stochasticity.
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- The real world is complicated.
  - Hard to collect data.
  - Many confounding complications.
  - Demographic stochasticity.

- The nature of science is more easily discovered using real data from an artificial world. (e.g., C.S. Holling, 1959)
  - Easy to collect data.
  - Based on simple mechanisms.
  - Must have demographic stochasticity!
Experiential – Materials and Setup

X — square is occupied
○ — square is available

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Experiential – Simulation Rules

1. For each available square:
   a. Roll one die for each adjacent occupied square.
   b. If any die is 5 or 6, mark the square with a slash (/).
2. Change the slashes into X’s. Record population.
3. Mark new available squares with a circle (◦).

Stop when nearly all squares are occupied.

X — square is occupied
◦ — square is available
### Numerical – Lots of Data

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### Verbal – Presentation

- An Experiential Representation
- A Numerical Representation
- Three Visual Representations
- A Verbal Representation
- A Symbolic Representation

### Acknowledgements

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Acknowledgements
Visual – Population Graphs
Visual – What Else?

- Can we think of other, possibly better, ways to plot the data?
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  - Notice that slopes of the orange and green lines are the same for the same populations?
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- How about plotting population change vs population?
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- How about plotting population change vs population?

- Other ideas?
Visual – Change vs Population
Visual – Relative Change vs Population

- Relative change (\(\frac{dP}{P}\)) vs population
- Plot shows a decreasing trend as population increases.
Verbal – An Empirical Hypothesis

- Ignore the demographic stochasticity (scatter).
  - Is there a signal hiding under the noise?
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- Maybe the relative change is a linear function of the population, reaching 0 when the space is full.
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Symbolic – Dynamic Equation

- **Discrete**
  \[
  \frac{\Delta P}{P} = r \left(1 - \frac{P}{K}\right)
  \]
  or
  \[
  \Delta P = rP \left(1 - \frac{P}{K}\right)
  \]

- **Continuous**
  \[
  \frac{dP}{dt} = r \left(1 - \frac{P}{K}\right)
  \]
  or
  \[
  \frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right)
  \]

We need numerical implementation of a statistical method to fit \( r \) to the data (given \( K \)).

(See Ledder, Coll Math J, 47 (109), 2017.)
Improvements

- So far, we’re working with very limited data (like real ecologists) and a very simple setting. With a computer simulation, we can add detail and collect much more data quickly.
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PopGrowth.R (GL) and LogGrowth.nlogo (M.D. LaMar)

- arena size, $8 \leq s \leq 50$, best at about 20
- birth probability, $0 < b < 1$, best at 0.1 to 0.8
- death probability, must be $0 \leq d < b$, best at 0 to $b/4$
- number of trials, 1 to 4
- starting setup: center or edge
- curve fit options: none, $r$ only, $r$ and $K$
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- In this study, we developed a model for a synthetic system.
  - We actually know the true biological processes, which are completely different from the model.

- But the model does a great job of predicting the results.
Acknowledgement

- Special thanks to Carrie for her kind offer to share her talk slot with someone whose first priority had to be giving a research talk.
Acknowledgements

- Thank you to SMB for the day of education events.
- For more information, see our work here:

A Framework for Modeling to Encourage Interdisciplinary Conversations

Dr. Carrie Diaz Eaton
Dr. Hannah Callender
Dr. Kam Dahlquist
Dr. Drew LaMar
Dr. Glenn Ledder
Dr. Richard Schugart

Special issue on Interdisciplinary Conversations in PRIMUS

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