

Lab Assignment 4

Discrete-Time Food-Web Models

Written by

Prof. Erin N. Bodine

Prepared for

Math 214: Discrete Math Modeling (Fall 2014)

Rhodes College

PreLab Assignment Due: **Thursday, 9/25/2014 before class begins**

Lab Assignment Due: **Tuesday, 9/30/2014 before class begins**

Material from the prelab is covered on the quiz on **9/25/2014**.

In the reference readings we examined a model for a predator-prey system,

$$x_{n+1} = rx_n - \alpha x_n y_n \quad y_{n+1} = sy_n + \beta x_n y_n, \quad (1)$$

where x_n is the density of the prey, y_n is the density of the predators, $r > 1$ is the growth rate of the prey in the absence of the predator population, $0 < s < 1$ is the survival rate of the predator population in the absence of its prey source, α is the consumption rate of the predators (where αx_n is the average number of prey eaten per predator in time step n), and $\beta > 0$ is the growth rate of the predator population due to the consumption of prey.

A predator-prey systems is an example of an simple food-web containing two species in an ecosystem. However, food-webs or food-chains typically involve more than one species. In this lab project we will consider two alternative food-webs each containing three species (see Figures 1b and 1c).

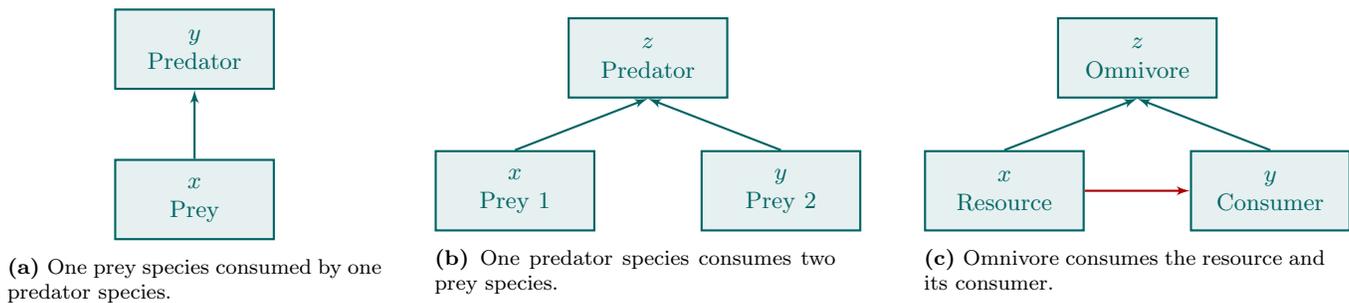


Figure 1: Three examples of food-web models. Each arrow points to the species which does the consuming.

The goals of this lab assignment are to:

- Analyze and simulate two different food-web models each containing three species, and compare the results of the two different models.
- Visualize the relationship between equilibrium points and time-series solutions.

In this lab, we will be using MATLAB to help achieve these goals. Thus, as a part of this lab you will learn how to do the following in MATLAB:

- Construct a phase-portraits.

1 MATLAB Skills

After completing Labs 1-3 you have most of the skill necessary to complete this lab project. In previous labs you have constructed and plotted the time-series solutions for a single difference equation. Note, a time-series solution refers to the sequence of values generated by a difference equation, and a time-series plot is a graph of that sequence of values over time (i.e., the horizontal axis represent the sequence index t or n , and the vertical axis represents the value of the terms in the sequence x_t or x_n). In this lab, you will construct and plot time-series solutions for systems of difference equations.

In addition to the time-series solutions, you will also plot phase portraits. A phase-portrait shows the time-series solution in the state space of a system of difference equations. If, for example, your system of difference equations generates sequences for x_n and y_n , then the state space of the system is (x_n, y_n) , and the phase portrait would plot

the time-series solution in the xy -plane. If your system of difference equations generates sequences for x_n , y_n , and z_n , then the state space of the system is (x_n, y_n, z_n) . For this system phase portraits can be generated in the xyz space, but it is more typical to examine phase portraits in the xy -, xz -, and yz -planes. In this lab, we will do the latter.

Plots of Systems of Difference Equations

1. Open MATLAB and navigate the current directory to the location where you will save your work.
2. From Moodle, download the script file L04Ex01.m.

L04Ex01.m

```
1 % Lab 4 PreLab by Prof Bodine
2 % Simulates Predator-Prey Model
3 clear all
4
5 %% Parameters
6 r = 1.0015;
7 s = 0.9994;
8 alpha = 0.0006;
9 beta = 0.00025;
10
11 x0 = 2;
12 y0 = 3;
13
14 years = 100;
15 T = 365*years; % Each time step represents 1 day
16
17 %% Generate Solutions over T years
18 x(1) = x0; y(1) = y0;
19 for t = 1:T
20     x(t+1) = r*x(t) - alpha*x(t)*y(t);
21     y(t+1) = s*y(t) + beta *x(t)*y(t);
22 end
23
24 %% Graphs of Solutions
25
26 % Plot time-series solution over time
27 time = 0:years/T:years;
28 subplot(1,2,1); plot(time,x,'r-',time,y,'b-')
29 xlabel('Time (years)')
30 ylabel('Population Density')
31 legend('x','y')
32
```

```

33 % Plot phase portrait in xy plane
34 subplot(1,2,2); plot(x,y,'c'); hold on;
35 plot(x0,y0,'kh'); % Plot Initial Condition
36 plot((1-s)/beta,(r-1)/alpha,'mp') ; % Plot Non-trivial equilibrium
37 hold off;
38 xlabel('x'); ylabel('y');

```

3. Now, let us take a moment to understand everything contained within this script.

- The first three lines give a description of the file in comments and use the `clear` command to erase the values of all variables and parameters from the workspace. If you do not remember how this command works, please review Lab 2, page 3.
- Lines 5–12 define the values of the parameters of the model. Notice that in lines 11 and 12 we define the initial conditions of the system, and there appears to be two state variables, x and y .
- Lines 14–15 define the number of time steps in the simulation. Since each time step in the model will represent 1 day, the total number of time steps is 365 times the number of years over which we wish to simulate.
- In lines 17–22, the solution of the system of difference equations is generated. Notice that in each pass through the loop generates values two two sequences,

$$x_{t+1} = rx_t - \alpha x_t y_t \quad \text{and} \quad y_{t+1} = sy_t + \beta x_t y_t,$$

which is the same system presented in Equation (1). Thus, this code is simulating the predator-prey model discussed in the reference readings.

Note that line 18 sets the initial condition for both the x sequence and the y sequence. If you wish to include two commands on one line, you must place a semi-colon after the first command.

- Lines 26–31 creates a graph of the time-series solution over time. Recall that each time step represents one day. If we used the command

```
plot(0:T,x,'r-', 0:T,y,'b-')
```

we would be plotting over $[0, 36500]$. This makes it difficult for someone looking at the graph to interpret the results, so we rescale the horizontal axis to represent years (see line 27). The code `0:years/T:years` creates an array from 0 to `years` (which was defined to be 100 in line 14) in step sizes of `years/T` (which calculates to $\frac{1}{365}$). Note, this array has 36501 elements, which is the same as the arrays `x` and `y`.

Notice in line 28, that both sequences (`x` and `y`) are plot in one of two subplots.

- Lines 33–38 creates the second subplot which is a phase portrait of the solution. Line 34 generates the plot of the sequences of `x` and `y` in the xy -plane in cyan. Line 35 plots a black hexagonal star at the initial point (x_0, y_0) , and line 36 plots a magenta pentagonal star at the non-trivial equilibrium of the system. Recall from the reference readings that the predatory-prey system defined by Equation (1) had two equilibria: $(0, 0)$ the trivial equilibrium, and $\left(\frac{1-s}{\beta}, \frac{r-1}{\alpha}\right)$ the single non-trivial equilibrium.

- (g) Take another moment to review L04Ex01.m. **Do you understand what each line of code is doing?** If not, now is the time to stop and ask.

4. Run the script file. The resulting graphs should look like Figure 2

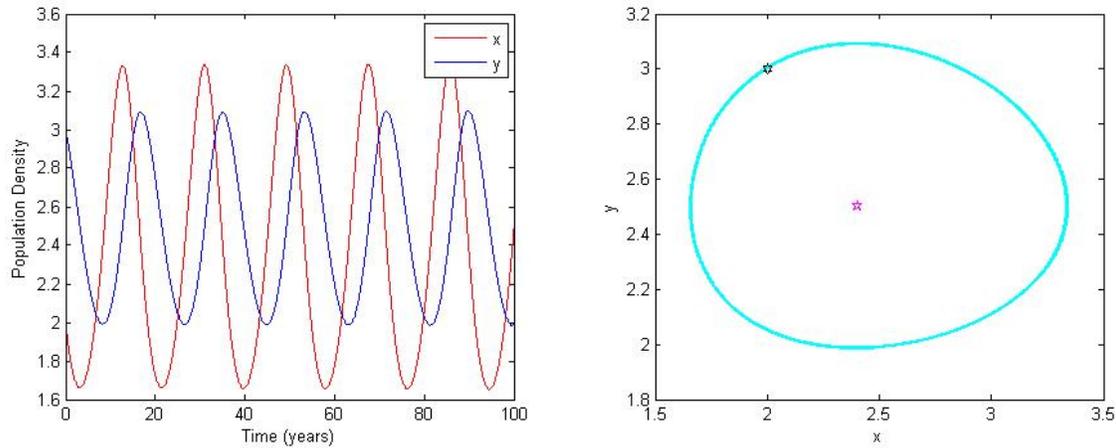


Figure 2: Plot produced from L04Ex01.m script.

Notice that each population is oscillating over time, which results in the phase portrait producing an egg-like shape that is traced over and over each time the oscillation repeats. Additionally, notice that in the phase portrait the egg-like shape is “centered” about the non-trivial equilibrium. We can describe this equilibrium point as being a *center* since it is the center of some stable oscillation. This would also imply that the other equilibrium, the trivial equilibrium, is unstable.

2 PreLab Assignment

Santa Cruz island is located off the coast of California near Santa Barbara. The Santa Cruz island fox (*Urocyon littoralis santacruzae*) is a dwarfed version of the mainland gray fox, and has evolved on the island over the past 10,000 years. More recently, in the 1850s, domestic pigs were introduced to the island. Over time, a large population of feral (non-domestic) pigs developed. In the 1980s a small population of golden eagles (*Aquila chrysaetos*) established territory on Santa Cruz island, prey on both the island foxes and the feral pigs. Let us consider a system of difference equations which models one predatory with two prey sources (see Figure 1b).



(a) Santa Cruz island fox



(b) Golden eagle



(c) Feral pig

Figure 3: Species in a food-web on Santa Cruz island.

Let x_n be the density of the island fox population at time step n , y_n be the density of the feral pig population at time step n , and z_n be the density of the golden eagle population at time step n . Then the food-web system can be modeled by

$$x_{t+1} = rx_t - \alpha x_t z_t, \quad y_{t+1} = sy_t - \beta y_t z_t, \quad z_{t+1} = dz_t + \gamma x_t z_t + \sigma y_t z_t, \quad (2)$$

where $r > 1$ is the growth rate of the island fox in the absence of the golden eagles, $s > 1$ is the growth rate of the feral pigs in the absence of the golden eagles, $0 < d < 1$ is the survival rate of the golden eagles in the absence of any prey source, $\alpha > 0$ and $\beta > 0$ are the golden eagle consumption rates of the island foxes and feral pigs, respectively (where αx_n is the average number of island foxes eaten per golden eagle per time step, and βy_n is the average number of feral pigs eat per golden eagle per time step), and γ and σ are the growths rates of the golden eagle population due to consumption of the island foxes and feral pigs, respectively.

1. Find the equilibrium points of System (2). *Hint: There are three equilibria: the trivial equilibrium, one of the form $(x^*, 0, z^*)$, and one of the form $(0, y^*, z^*)$.*
2. Write a MATLAB script similar to L04Ex01.m to simulate the model in Equation (2) for 100 years. Save your script file as `PLab04_YourLastName.m`. Use the parameter values $r = 1.0015$, $s = 1.0020$, $d = 0.9994$, $\alpha = 0.00025$, $\beta = 0.00035$, $\gamma = 0.00011$, $\sigma = 0.00014$, $x_0 = 2$, $y_0 = 2.5$, $z_0 = 3$, and a single time step represents 1 day.

Produce time series graph and phase portraits in the xz - and yz -planes. For the phase portrait in the xz -plane, plot the location of the equilibrium of the form $(x^*, 0, z^*)$. For the phase portrait in the yz -plane, plot the location of the equilibrium of the form $(0, y^*, z^*)$. All three graphs should be produced in one figure. Save the figure produced from this simulation as `PLab04A_YourLastName.jpg`. Interpret the graphs to describe what is happening to the three populations over time.

3. For the above simulation, change to simulate over 500 years. Save the figure produced from this simulation as `PLab04B_YourLastName.jpg`. Describe what is happening to the three populations over time. Classify each of the three equilibria as either stable, unstable, or a center.
4. Set $\alpha = 0.00036$ and run the simulation for 100 years. Save the figure produced from this simulation as `PLab04C_YourLastName.jpg`. Interpret what is happening to the three populations over time and compare to the previous simulation.
5. Submit your script file, three image files, and answers to all questions posed in parts 1-4. Submit your written answer in the provided text box.

3 Lab Assignment

Some food-webs are more complex than one predator having two prey. Imagine food-web of owls, snakes, and mice (or voles, or rats, or the small rodent of your choice). Many species of medium to large sized owls can feed on both snakes and small rodents, however, the small rodents are also a food source for the snakes. Though there are still only three species, the food-web is now more complicated.

Let x_n represent the density of the rodents at time step n , y_n the density of the snakes at time step n , and z_n the

density of the owls at time step n . Then the food-web system can be modeled by

$$x_{t+1} = rx_t - \alpha x_t y_t - \beta x_t z_t, \quad y_{t+1} = sy_t + \gamma x_t y_t - \delta y_t z_t, \quad z_{t+1} = dz_t + \sigma x_t z_t + \theta y_t z_t, \quad (3)$$

where $r > 1$ is the growth rate of the rodents in the absence of both owls and snakes, $0 < s < 1$ is the survival rate of the snakes in the absence of their food source (rodents), $0 < d < 1$ is the survival rate of the owls in the absence of both food sources (rodents and snakes), $\alpha, \beta, \delta > 0$ are the consumption rates of the rodents by the snakes, the rodents by the owls, and the owls by the snakes, respectively, and $\gamma, \sigma, \theta > 0$ are the growth rates of the snakes due to consumption of the rodents, owls due to consumption of the rodents, and owls due to consumption of the snakes, respectively.

1. There are five equilibrium points of System (3): (1) the trivial equilibrium, (2) an equilibrium of the form $(x^*, y^*, 0)$, (3) an equilibrium of the form $(x^*, 0, z^*)$, (4) an equilibrium of the form $(0, y^*, z^*)$, and (5) an equilibrium of the form (x^*, y^*, z^*) . Find expressions for equilibria (2), (3), and (4) and explain why equilibria (4) is never biologically reasonable.
2. Write a MATLAB script similar to `L04Ex01.m` to simulate the model in Equation (3) for 200 years. Save your script file as `Lab04_YourLastName.m`. Use the parameter values $r = 1.0015$, $s = 0.9996$, $d = 0.9994$, $\alpha = 0.00015$, $\beta = 0.00015$, $\gamma = 0.00025$, $\delta = 0.00015$, $\sigma = 0.00015$, $\theta = 0.00010$, $x_0 = 2$, $y_0 = 2.5$, $z_0 = 3$, and a single time step represents 1 day.

Produce phase portraits in the xz - and xy -planes. For the phase portrait in the xz -plane, plot the location of the equilibrium of the form $(x^*, 0, z^*)$. For the phase portrait in the xy -plane, plot the location of the equilibrium of the form $(x^*, y^*, 0)$. All three graphs should be produced in one figure. Save the figure produced from this simulation at `Lab04A_YourLastName.jpg`. Interpret the graphs to describe what is happening to the three populations over time. Classify each equilibrium (1), (2), and (3) as either stable, unstable, or a center.

3. Set $d = 0.9985$ and run the simulation for 200 years. Save the figure produced from this simulation as `Lab04B_YourLastName.jpg`. Interpret the graphs to describe what is happening to the three populations over time and compare to the previous simulation. Classify each equilibrium (1), (2), and (3) as either stable, unstable, or a center.

Note, in each of the above two simulations equilibrium (5) is biologically unreasonable (i.e. has at least one negative component).

4. Think of a food web that has four species. Give a quick description of the biology of this food-web, propose a system of difference equations to model your proposed food-web, and describe each of the parameters in your model. You do not have to simulate your model in MATLAB.

5. Submit:

- 1 script file (`Lab04_YourLastName.m`)
- 2 image files (`Lab04A_YourLastName.jpg` and `Lab04B_YourLastName.jpg`)
- Answers to all questions posed in parts 1-4.