ABOUT THE LAB:

We will solve the mathematical equations describing the simple pendulum using Matlab. We will compare the numerical solutions to exact solutions (using the small-angle approximation) and solutions resulting from different numerical methods.

Exercise 1. The equation of motion for a simple pendulum is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

where $\theta(t)$ is the angular displacement, L is the length of the arm, and g is the acceleration due to gravity.
(a) Use the Taylor approximation for $\sin\theta$ near $\theta_0 = 0$, to show that, for small angles, the differential equation can be written as

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta.$$

(b) Show that $\theta(t) = c_1\cos(\sqrt{\frac{g}{L}} + c_2)$ is a solution of the equation in part (a). Here the two constants, c_1 and c_2 are determined by the initial conditions (initial angular displacement and initial velocity).
(c) The equation for the simple pendulum can also be written as

$$\begin{aligned} \frac{d\omega}{dt} &= -\frac{g}{L}\sin\theta \\ \frac{d\theta}{dt} &= \omega. \end{aligned}$$

Explain what ω represents.
(d) The Euler method for solving this system of differential equations can be written as

$$\theta_{n+1} = \theta_n + h\omega_n \qquad (1)$$

$$\omega_{n+1} = \omega_n - h\frac{g}{L}\sin\theta_n \qquad (2)$$

Implement the Euler method to solve this system.
(e) Run the simulation using an initial $\theta(0) = 10$ (in degrees), with time step $h = 0.1$ for 300 time steps. Plot the solution and describe what you see.
(f) On the same plot, show what happens if you decrease the time step to $h = 0.01$, then $h = 0.001$. (Run all of the simulations to the same length of time.) Also plot the analytical solution for small angles (shown in part (b)). Compare the four solutions.
(g) (COMPLETE THIS AND REMAINING PARTS IF THERE IS TIME) Now use a built-in Matlab solver to solve Eqns 1-2. For a template, you can use the Matlab file for the concentration of a drug in the body from Lab 1.

(h) Compare the numerical solution using the built-in Matlab solver to your analytical solution and the numerical solution using the Euler method.

(i) EXTRA CREDIT

Conduct a numerical experiment to find out how large you can make the initial angular displacement and still use the small-angle approximation with good accuracy. Do this by running a number of simulations (using the original equations), each with a larger initial angle. Use the built-in Matlab solver. To determine whether your approximation is good, compute the norm of the difference between the small-angle analytical solution and the numerical solution. This will tell you the size of the error between the numerical and analytical solutions. Stop the process when the error exceeds 10% of the maximum value of the angular displacement. What is the largest angle you found for which the small-angle approximation worked?